

On continued fraction representations of certain function of hypergeometric type

S. Ahmad Ali and *Syed Siddique Mustafa Rizvi,

Department of Mathematics,

BBD University, Lucknow (U.P.) India.

E-mail- ahmad67@rediffmail.com

* Department of Mathematics,

RR Institute of Modern Technology, Lucknow (U.P.) India.

E-mail: ssm.rizvi@gmail.com

(Received November 09, 2012)

Abstract: In the present work we have established a number of continued fraction representations for the function $F(a, b, q)$ which is a special case of Heine function. Our results produce certain new and many known results in the literature as their special cases.

Keywords: Heine's function, continued fraction, mock theta function, Rogers-Ramanujan identity.

AMS subject classification code: 33D15, 11A55

(A) Introduction: Let, for $|q| < 1$

$$(a; q)_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1})$$

and

$$(a; q)_0 = 1$$

Then Heine series has been defined as,

$${}_2\Phi_1[\alpha, \beta; \gamma; z] = \sum_{n=0}^{\infty} \frac{(\alpha; q)_n (\beta; q)_n z^n}{(q; q)_n (\gamma; q)_n}$$

Taking $\alpha = aq$, $\beta = q$ & $\gamma = bq$,

$$F(a, b, z) = {}_2\Phi_1[aq, q; bq; z] = \sum_{n=0}^{\infty} \frac{(aq)_n}{(bq)_n} z^n$$

The function $F(a, b, z)$ has been extensively studied by Fine [1]. It is easy to see that $F(a, b, z)$ satisfies the following relations.

$$F(a, b, t) = \frac{(1 - atq)}{(1 - t)} + \frac{(1 - aq)(b - atq)}{(1 - bq)(1 - t)} tq F(aq, bq, tq) \quad (1.1)$$

$$F(a, b, t) = \frac{1}{(1 - t)} + \frac{(b - a)tq}{(1 - bq)(1 - t)} F(a, bq, tq) \quad (1.2)$$

$$F(a, b, t) = \frac{b}{(b - at)} + \frac{(b - a)t}{(1 - bq)(b - at)} F(a, bq, t) \quad (1.3)$$

$$F(a, b, t) = \frac{-(1 - b)aq}{b - aq} + \frac{(1 - aq)(b - atq)}{(b - aq)} F(aq, b, t) \quad (1.4)$$

In the present paper we have established many continued fraction representation for $F(a, b, z)$ by making use of relations (1), (2), (3) and (4).

As an application of our results obtained in section -B, we have a number of special cases in section -C.

(B) Main Results: In this section, we establish the following results,

$$\frac{F(a, b, t)}{F(aq, bq, tq)} = A_0 + \frac{B_0}{A_1 + \frac{B_1}{A_2 + \frac{B_2}{A_3 + \dots}}} \quad (1.5)$$

where

$$\begin{aligned} A_i &= \frac{1}{(1 - t)} \left[\frac{(1 - aq^{i+1})(b - atq^{i+1})tq^{2i+1}}{(1 - bq^{i+1})} + \frac{(1 - atq^{2i+1})(1 - tq^{i+1})}{(1 - atq^{2i+3})} \right] \\ B_i &= \frac{(aq^{i+2} - 1)(b - atq^{i+2})(1 - atq^{2i+1})tq^{i+2}}{(1 - tq^i)(1 - bq^{i+2})(1 - atq^{2i+2})} \\ \frac{F(a, b, t)}{F(a, bq, tq)} &= A_0 + \frac{B_0}{A_1 + \frac{B_1}{A_2 + \frac{B_2}{A_3 + \dots}}} \end{aligned} \quad (1.6)$$

where

$$A_i = \frac{(1 - tq^{i+1})}{(1 - tq^i)} \left[1 + \frac{(bq^i - a)tq^{i+1}}{(1 - bq^{i+1})(1 - tq^i)(1 - tq^{i+1})} \right]$$

$$B_i = \frac{(a - bq^{i+1})tq^{i+2}}{(1 - tq^i)(1 - bq^{i+2})}$$

$$\frac{F(a, b, t)}{F(a, bq, t)} = A_0 + \frac{B_0}{A_1 + \frac{B_1}{A_2 + \frac{B_2}{A_3 + \dots}}} \quad (1.7)$$

where

$$A_i = \frac{1}{(bq^i - at)} \left[\frac{(bq^i - a)t}{(1 - bq^{i+1})} + \frac{(bq^{i+1} - at)}{q} \right]$$

$$B_i = \frac{(a - bq^{i+1})t}{q(bq^i - at)(1 - bq^{i+2})}$$

$$\frac{F(a, b, t)}{F(aq, b, t)} = A_0 + \frac{B_0}{A_1 + \frac{B_1}{A_2 + \frac{B_2}{A_3 + \dots}}} \quad (1.8)$$

$$A_i = \frac{1}{(b - aq^{i+1})} \left[(1 - aq^{i+1})(b - atq^{i+1}) + \frac{(b - aq^{i+2})}{q} \right]$$

$$B_i = \frac{(aq^{i+2} - 1)(b - atq^{i+2})}{q(b - aq^{i+1})}$$

Proof of (1.5)

$$F(a, b, t) = \frac{(1 - atq)}{1 - t} + \frac{(1 - aq)(b - atq)}{(1 - bq)(1 - t)} tq F(aq, bq, tq)$$

$$F(aq, bq, tq) = \frac{(1 - atq^3)}{(1 - tq)} + \frac{(1 - at^2)(bq - atq^3)}{(1 - bq^2)(1 - tq)} tq^2 F(aq^2, bq^2, tq^2)$$

$$\frac{(1 - atq^2)}{(1 - tq)} F(a, b, t) = \left[\frac{(1 - atq^3)(1 - aq)(b - atq)}{(1 - tq)(1 - bq)(1 - t)} tq + \frac{(1 - atq)}{1 - t} \right] F(aq, bq, tq)$$

$$- \frac{(1 - aq^2)(bq - atq^2)(1 - atq)}{(1 - bq^2)(1 - tq)(1 - t)} tq^2 F(aq^2, bq^2, tq^2)$$

$$\frac{(a, b, t)}{F(aq, bq, tq)} = \left[\frac{(1 - aq)(b - atq)}{(1 - bq)} tq + \frac{(1 - atq)(1 - tq)}{(1 - atq^2)} \right] \frac{1}{(1 - t)}$$

$$- \frac{(1 - aq^2)(b - atq^2)(1 - atq)tq^2}{(1 - t)(1 - bq^2)(1 - atq^2)} \frac{F(aq, bq, tq)}{F(aq^2, bq^2, tq^2)}$$

$$\frac{F(a, b, t)}{F(aq, bq, tq)} = A_0 + \frac{B_0}{A_1 + \frac{B_1}{A_2 + \frac{B_2}{A_3 + \dots}}}$$

where

$$A_i = \frac{1}{(1-t)} \left[\frac{(1-aq^{i+1})(b-atq^{i+1})tq^{2i+1}}{(1-bq^{i+1})} + \frac{(1-atq^{2i+1})(1-tq^{i+1})}{(1-atq^{2i+2})} \right]$$

$$B_i = \frac{(aq^{i+2}-1)(b-atq^{i+2})(1-atq^{2i+1})tq^{i+2}}{(1-tq^i)(1-bq^{i+2})(1-atq^{2i+2})}$$

Proof of (1.6)

$$F(a, b, t) = \frac{1}{(1-t)} + \frac{(b-a)tq}{(1-bq)(1-t)} F(a, bq, tq)$$

$$F(a, bq, tq) = \frac{1}{(1-tq)} + \frac{(bq-a)tq^2}{(1-bq^2)(1-tq)} F(a, bq^2, tq^2)$$

$$\frac{1}{(1-tq)} F(a, b, t) = \frac{F(a, bq, tq)}{(1-t)} \left[1 + \frac{(b-a)tq}{(1-bq)(1-t)(1-tq)} \right]$$

$$- \frac{(bq-a)tq^2}{(1-t)(1-bq^2)(1-tq)} F(a, bq^2, tq^2)$$

$$\frac{F(a, b, t)}{F(a, bq, tq)} = \frac{(1-tq)}{(1-t)} \left[1 + \frac{(b-a)tq}{(1-bq)(1-t)(1-tq)} \right]$$

$$- \frac{(bq-a)tq^2}{(1-t)(1-bq^2)} \frac{F(a, bq, tq)}{F(a, bq^2, tq^2)}$$

$$\frac{F(a, bq, tq)}{F(a, bq^2, tq^2)} = \frac{(1-tq^2)}{(1-t)} \left[1 + \frac{(b-a)tq^2}{(1-bq^2)(1-tq)(1-tq^2)} \right]$$

$$- \frac{(bq^2-a)tq^3}{(1-tq)(1-bq^3)} \frac{F(a, bq^2, tq^2)}{F(a, bq^3, tq^3)}$$

$$\frac{F(a, b, t)}{F(a, bq, tq)} = A_0 + \frac{B_0}{A_1 + \frac{B_1}{A_2 + \frac{B_2}{A_3 + \dots}}}$$

where

$$A_i = \frac{(1 - tq^{i+1})}{(1 - tq^i)} \left[1 + \frac{(bq^i - a)tq^{i+1}}{(1 - bq^{i+1})(1 - tq^i)(1 - tq^{i+1})} \right]$$

$$B_i = \frac{(a - bq^{i+1})tq^{i+2}}{(1 - tq^i)(1 - bq^{i+2})}$$

Proof of (1.7)

$$F(a, b, t) = \frac{b}{(b - at)} + \frac{(b - a)t}{(1 - bq)(b - at)} F(a, bq, t)$$

$$F(a, bq, t) = \frac{bq}{(bq - at)} + \frac{(bq - a)t}{(1 - bq^2)(bq - at)} F(a, bq^2, t)$$

$$\frac{F(a, b, t)}{F(a, bq, t)} = \left[\frac{(b - a)t}{(1 - bq)(b - at)} + \frac{(bq - at)}{q(b - at)} \right] - \frac{(bq - a)t}{q(b - at)(1 - bq^2)} \frac{F(a, bq, t)}{F(a, bq^2, t)}$$

$$\frac{F(a, bq, t)}{F(a, bq^2, t)} = \left[\frac{(bq - a)t}{(1 - bq^2)(bq - at)} + \frac{(bq^2 - at)}{q(b - at)} \right] - \frac{(bq^2 - a)t}{q(bq - at)(1 - bq^3)} \frac{F(a, bq^2, t)}{F(a, bq^3, t)}$$

$$\frac{F(a, b, t)}{F(a, bq, t)} = A_0 + \frac{B_0}{A_1 + \frac{B_1}{A_2 + \frac{B_2}{A_3 + \dots}}}$$

where

$$A_i = \frac{1}{(bq^i - at)} \left[\frac{(bq^i - a)t}{(1 - bq^{i+1})} + \frac{(bq^{i+1} - at)}{q} \right]$$

$$B_i = \frac{(a - bq^{i+1})t}{q(bq^i - at)(1 - bq^{i+2})}$$

Proof of (1.8)

$$F(a, b, t) = \frac{-(1 - b)aq}{(b - aq)} + \frac{(1 - aq)(b - atq)}{(b - aq)} F(aq, b, t)$$

$$F(aq, b, t) = \frac{-(1 - b)aq^2}{(b - aq^2)} + \frac{(1 - aq^2)(b - atq^2)}{(b - aq^2)} F(aq^2, b, t)$$

$$\frac{qF(a, b, t)}{(b - aq^2)} = \left[\frac{(1 - aq)(b - atq)q}{(b - aq)(b - aq^2)} + \frac{1}{(b - aq)} \right] F(aq, b, t)$$

$$\begin{aligned}
& -\frac{(1-aq^2)(b-atq^2)}{(b-aq^2)(b-aq)}F(aq^2, b, t) \\
\frac{F(a, b, t)}{F(aq, b, t)} &= \left[\frac{(1-aq)(b-atq)}{(b-aq)} + \frac{(b-aq^2)}{q(b-aq)} \right] - \frac{(1-aq^2)(b-atq^2)}{q(b-aq)} \frac{F(aq, b, t)}{F(aq^2, b, t)} \\
\frac{F(aq, b, t)}{F(aq^2, b, t)} &= \left[\frac{(1-aq^2)(b-atq^2)}{(b-aq^2)} + \frac{(b-aq^3)}{q(b-aq^2)} \right] - \frac{(1-aq^3)(b-atq^3)}{q(b-aq^2)} \frac{F(aq^2, b, t)}{F(aq^3, b, t)} \\
\frac{F(a, b, t)}{F(aq, b, t)} &= A_0 + \frac{B_0}{A_1 + \frac{B_1}{A_2 + \frac{B_2}{A_3 + \dots}}}
\end{aligned}$$

where

$$\begin{aligned}
A_i &= \frac{1}{(b-aq^{i+1})} \left[(1-aq^{i+1})(b-atq^{i+1}) + \frac{(b-aq^{i+2})}{q} \right] \\
B_i &= \frac{(aq^{i+2}-1)(b-atq^{i+2})}{q(b-aq^{i+1})}
\end{aligned}$$

(C) Applications (Special cases)

From equation (1.4), we have,

$$\begin{aligned}
F(a, b, t) &= \frac{-(1-b)aq}{(b-aq)} + \frac{(1-aq)(b-atq)}{(b-aq)}F(aq, b, t) \\
\frac{-(1-b)aq}{(b-aq)F(aq, b, t)} + \frac{(1-aq)(b-atq)}{(b-aq)} &= \frac{F(a, b, t)}{F(aq, b, t)} = A \quad \text{Say} \\
\frac{-(1-b)aq}{(b-aq)F(aq, b, t)} &= A - \frac{(1-aq)(b-atq)}{(b-aq)} \\
\frac{-(1-b)aq}{(b-aq)F(aq, b, t)} &= \frac{A(b-aq) - (1-aq)(b-atq)}{(b-aq)} \\
\frac{F(aq, b, t)}{-(1-b)aq} &= \frac{1}{\frac{A(b-aq) - (1-aq)(b-atq)}{(b-aq)} + A}
\end{aligned}$$

From equation (1.8), we have,

$$\frac{(b-aq)}{-(1-b)aq} F(aq, b, t) = \frac{1}{\frac{-(1-aq)(b-atq)}{(b-aq)} + A_0 + \frac{B_0}{A_1 + \frac{B_1}{A_2 + \frac{B_2}{A_3 + \dots}}}}$$

Putting $b=0$ in above we get

$$F(aq, 0, t) = \frac{1}{-(1-aq)t + X_0 + \frac{Y_0}{X_1 + \frac{Y_1}{X_2 + \frac{Y_2}{X_3 + \dots}}}}$$

$$X_i = (1 - aq^{i+1})t + 1$$

$$Y_i = (aq^{i+2} - 1)t$$

Now set $a = \frac{b}{tq}$ to get,

$$F(b/t, 0, t) = \frac{1}{-(1-b/t)t + P_0 + \frac{Q_0}{P_1 + \frac{Q_1}{P_2 + \frac{Q_2}{P_3 + \dots}}}}$$

$$P_i = (t - bq^i) + 1$$

$$Q_i = (bq^{i+1} - t)$$

From N.J. Fine [7.3(i)], we have

$$\frac{1}{1-t} \sum \frac{(-b)^n}{(tq)_n} q^{\frac{(n^2+n)}{2}} = \frac{1}{-(t-b) + P_0 + \frac{Q_0}{P_1 + \frac{Q_1}{P_2 + \frac{Q_2}{P_3 + \dots}}}}$$

Let $t \rightarrow 0$, in above,

$$\sum_{n \geq 0} (-b)^n q^{\frac{(n^2+n)}{2}} = \frac{1}{b + P_0 + \frac{Q_0}{P_1 + \frac{Q_1}{P_2 + \frac{Q_2}{P_3 + \dots}}}} \quad (1.9)$$

$$P_i = (1 - bq^i)$$

$$Q_i = bq^{i+1}$$

Case (a)

For $b = -1$, in equation (1.9), yields

$$\sum_{n \geq 0} q^{\frac{(n^2+n)}{2}} = \frac{1}{\frac{-q}{1 + q + \frac{-q^2}{1 + q^2 + \frac{-q^3}{1 + q^3 + \frac{-q^4}{1 + q^4 + \dots}}}}}$$

Now we write, by N.J. Fine [7.32(i)]

$$\sum_{n \geq 0} q^{\frac{(n^2+n)}{2}} = \prod_{n \geq 1} \frac{(1 - q^{2n})}{(1 - q^{2n-1})}$$

Hence we can write,

$$\prod_{n \geq 1} \frac{(1 - q^{2n})}{(1 - q^{2n-1})} = \frac{1}{\frac{-q}{1 + q + \frac{-q^2}{1 + q^2 + \frac{-q^3}{1 + q^3 + \frac{-q^4}{1 + q^4 + \dots}}}}}$$

This is a required result, equating a product to a continued fraction.

Case (b)

For $b = 1$, in equation (1.9), yields

$$\sum_{n \geq 0} (-1)^n q^{\frac{(n^2+n)}{2}} = \frac{1}{2 + \frac{q}{1 - q + \frac{q^2}{1 - q^2 + \frac{q^3}{1 - q^3 + \dots}}}}$$

Also, by N.J. Fine [7.32(ii)],

$$(q)_\infty^2 \sum_{n \geq 0} \frac{q^n}{(q)_n^2} = \frac{1}{2 + \frac{q}{1 - q + \frac{q^2}{1 - q^2 + \frac{q^3}{1 - q^3 + \dots}}}}$$

These are also required results, equating a sum to a continued fraction.

Case (c)

If we put $b = q^{-1/2}$ in equation (1.9) and then replacing $q \rightarrow q^2$,

$$\sum_{n \geq 0} (-1)^n q^{n^2} = \frac{1}{1 + \frac{q}{1 - q + \frac{q^3}{1 - q^3 + \frac{q^5}{1 - q^5 + \dots}}}}$$

Also by Auluck, we have,

$$\sum_{n \geq 0} \frac{q^{2n}}{(q)_{2n}} = \frac{1}{(q)_\infty} \sum_{n \geq 0} (-1)^n q^{n^2}$$

Hence we can write,

$$(q)_\infty \sum_{n \geq 0} \frac{q^{2n}}{(q)_{2n}} = \frac{1}{1 + \frac{q}{1 - q + \frac{q^3}{1 - q^3 + \frac{q^5}{1 - q^5 + \dots}}}}$$

If we put $b = q^{-1/2}$ in equation (1.9) and then replacing $q \rightarrow q^2$,

$$\sum_{n \geq 0} q^{n^2} = \frac{1}{1 + \frac{-q}{1 + q + \frac{-q^3}{1 + q^3 + \frac{-q^5}{1 + q^5 + \dots}}}}$$

Hence we obtained a continued fraction for a well known theta function.

References

- [1] Fine, N.J., Basic Hypergeometric Series and its Applications, Mathematical Survey & Monographs, No. 27, American Mathematical Society, 1988.