

A note on mock theta functions of order two

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Abstract: In this paper, making use of some known results, an attempt has been made to give continued fraction representations of mock theta functions of order two.

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1. Introduction:

Throughout this note, we shall adopt following notations and definitions. The q -shifted factorial is denoted by

$$[a; q]_n = (1 - a)(1 - aq) \dots (1 - aq^{n-1}), \quad n = 1, 2, 3 \dots$$

$$[a; q]_0 = 1,$$

$$[a; q]_\infty = \prod_{r=0}^{\infty} (1 - aq^r).$$

Basic hypergeometric series is defined by,

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r; q; z \end{matrix} \right] \\
= \sum_{n=0}^{\infty} \frac{[a_1, a_2, \dots, a_r; q]_n}{[b_1, b_2, \dots, b_s; q]_n} [(-)^n q^{n(n-1)/2}]^{1+s-r} z^n,$$

where $[a_1, a_2, \dots, a_r; q]_n = [a_1; q]_n [a_2; q]_n \dots [a_r; q]_n$.

Recently, McIntosh [1] introduced three second order mock theta functions as,

$$A(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2} [-q; q^2]_n}{[q; q^2]_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^{n+1} [-q^2; q^2]_n}{[q; q^2]_{n+1}}, \quad (1.1)$$

$$B(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)} [-q^2; q^2]_n}{[q; q^2]_{n+1}^2} = \sum_{n=0}^{\infty} \frac{q^n [-q; q^2]_n}{[q; q^2]_{n+1}}, \quad (1.2)$$

$$C(q) = \sum_{n=0}^{\infty} \frac{(-)^n q^{n^2} [q; q^2]_n}{[-q^2; q^2]_n^2}. \quad (1.3)$$

We can write these functions as

$$A(q) = \frac{q}{(1-q)^2} \lim_{\alpha \rightarrow \infty} {}_3\Phi_2 \left[\begin{matrix} \alpha, -q, q^2; q^2; -q^3/\alpha \end{matrix} \right], \quad (1.4)$$

$$= \frac{q}{(1-q)} {}_2\Phi_1 \left[\begin{matrix} -q^2, q^2; q^2; q \end{matrix} \right]. \quad (1.5)$$

$$B(q) = \frac{1}{(1-q)^2} \lim_{\alpha \rightarrow \infty} {}_3\Phi_2 \left[\begin{matrix} \alpha, -q^2, q^2; q^2; -q^2/\alpha \end{matrix} \right], \quad (1.6)$$

$$= \frac{1}{(1-q)} {}_2\Phi_1 \left[\begin{matrix} -q, q^2; q^2; q \end{matrix} \right]. \quad (1.7)$$

$$C(q) = \lim_{\alpha \rightarrow \infty} {}_3\Phi_2 \left[\begin{matrix} \alpha, q, q^2; q^2; q/\alpha \end{matrix} \right]. \quad (1.8)$$

We shall make use of following known results

$$\begin{aligned} \frac{{}_2\Phi_1 \left[\begin{matrix} \alpha, \beta q; z \end{matrix} \right]}{{}_2\Phi_1 \left[\begin{matrix} \alpha, \beta; z \end{matrix} \right]} &= \frac{1}{1 - \left(1 - \frac{\gamma}{\beta q}\right) +} \frac{\gamma(1 - \beta q) \left(1 - \frac{\alpha \beta z q}{\gamma}\right) / \beta q}{1 -} \\ &\frac{z \beta q (1 - \alpha q)}{\left(1 - \frac{\gamma}{\beta q}\right) +} \frac{\gamma(1 - \beta q^2) \left(1 - \frac{\alpha \beta z q^2}{\gamma}\right) / \beta q}{1 -} \frac{z \beta q^2 (1 - \alpha q^2)}{\left(1 - \frac{\gamma}{\beta q}\right) +} \dots \end{aligned} \quad (1.9)$$

[Singh, S.N. [3]]

$$\begin{aligned}
& \frac{{}_3\Phi_2 \left[\begin{matrix} a, b, c; \frac{de}{abc} \\ d, e \end{matrix} \right]}{{}_3\Phi_2 \left[\begin{matrix} aq, b, c; \frac{de}{abc} \\ dq, e \end{matrix} \right]} = 1 - \frac{\frac{de}{abc}(a-d)(1-b)(1-c)/(1-e)(1-d)(1-dq)}{\frac{(1-e/aq)}{(1-e)} +} \\
& \frac{(1-aq)(1-dq/b)(1-dq/c)\frac{e}{aq}/(1-e)(1-dq)(1-dq^2)}{1-} \\
& \frac{\frac{deq}{abc}(a-dq)(1-bq)(1-cq)/(1-eq)(1-dq^2)(1-dq^3)}{\frac{(1-e/aq)}{(1-e)} + \dots} \quad (1.10)
\end{aligned}$$

[Singh, S.N. [2]]

Putting $\beta = 1$ in (1.9) we find,

$$\begin{aligned}
& {}_2\Phi_1 \left[\begin{matrix} \alpha, q; z \\ \gamma \end{matrix} \right] = \frac{1}{1-} \frac{z(1-\alpha)}{(1-\gamma/q)+} \frac{\gamma(1-q) \left(1 - \frac{\alpha z q}{\gamma}\right) / q}{1-} \\
& \frac{zq(1-\alpha q)}{(1-\gamma/q)+} \frac{\gamma(1-q^2) \left(1 - \frac{\alpha z q^2}{\gamma}\right) / q}{1-} \frac{zq^2(1-\alpha q^2)}{(1-\gamma/q) + \dots} \quad (1.11)
\end{aligned}$$

Putting $a = 1$ in (1.10) we have,

$$\begin{aligned}
& {}_3\Phi_2 \left[\begin{matrix} b, c, q; \frac{de}{abc} \\ dq, e \end{matrix} \right] = \frac{1}{1-} \frac{\frac{de}{bc}(1-d)(1-b)(1-c)/(1-e)(1-d)(1-dq)}{\frac{(1-e/q)}{(1-e)} +} \\
& \frac{(1-q)(1-dq/b)(1-dq/c)\frac{e}{q}/(1-e)(1-dq)(1-dq^2)}{1-} \\
& \frac{\frac{deq}{bc}(1-dq)(1-bq)(1-cq)/(1-eq)(1-dq^2)(1-dq^3)}{\frac{(1-e/q)}{(1-e)} + \dots} \quad (1.12)
\end{aligned}$$

2. Main Results:

In this section we shall give continued fraction representation of mock theta functions of order two. **(i)** Replacing q by q^2 in (1.11) and then taking $\alpha = -q^2$ and $\gamma = q^3, z = q$ we get from the definition of $A(q)$ in (1.5),

$$A(q) = \left(\frac{q}{1-q} \right) \left\{ \frac{1}{1-} \frac{q(1+q^2)}{(1-q)+} \frac{q(1-q^4)}{1-} \frac{q^3(1+q^4)}{(1-q)+} \frac{q(1-q^8)}{1-} \dots \right\} \quad (2.1)$$

(ii) Replacing q by q^2 in (1.12) and then taking $b = \alpha, c = -q, d = q, e = q^3$ we get from the definition of $A(q)$ in (1.4)

$$\begin{aligned} A(q) &= \frac{q}{(1-q)^2} \left\{ \frac{1}{1+} \frac{\frac{q^3}{\alpha}(1-\alpha)(1+q)/(1-q^3)(1-q^3)}{\left(\frac{1-q}{1-q^3} \right) +} \right. \\ &\quad \frac{(1-q^2)(1-q^3/\alpha)(1+q^2)q/(1-q^3)(1-q^3)(1-q^5)}{1+} \\ &\quad \left. \frac{\frac{q^4}{\alpha}(1-\alpha q^2)(1+q^3)(1-q^3)/(1-q^5)(1-q^5)(1-q^7)}{\left(\frac{1-q}{1-q^5} \right) + \dots} \right\} \quad (2.2) \\ &= \lim_{\alpha \rightarrow \infty} \frac{q}{(1-q)^2} \left\{ \frac{1}{1+} \frac{q^3(1+q) \left(\frac{1}{\alpha} - 1 \right) / (1-q^3)^2}{\left(\frac{1-q}{1-q^3} \right) +} \frac{(1-q^4)(1-q^3/\alpha)q/(1-q^3)^2(1-q^5)}{1+} \right. \\ &\quad \left. \frac{q^4(1-q^6) \left(\frac{1}{\alpha} - q^2 \right) / (1-q^5)^2(1-q^7)}{\left(\frac{1-q}{1-q^5} \right) + \dots} \right\} \\ &= \frac{q}{(1-q)^2} \left\{ \frac{1}{1-} \frac{q^3(1+q)/(1-q^3)^2}{\left(\frac{1-q}{1-q^3} \right) +} \frac{q(1-q^4)/(1-q^3)^2(1-q^5)}{1-} \right. \\ &\quad \left. \frac{q^6(1-q^6)/(1-q^5)^2(1-q^7)}{\left(\frac{1-q}{1-q^5} \right) + \dots} \right\} \quad (2.3) \end{aligned}$$

$$A(q) = \frac{q}{(1-q)^2} \left\{ \frac{1}{1-} \frac{q^3(1+q)/(1-q^3)}{(1-q)+} \frac{q(1-q^4)/(1-q^3)(1-q^5)}{1-} \right. \\ \left. \frac{q^6(1-q^6)/(1-q^5)(1-q^7)}{(1-q)+\dots} \right\} \quad (2.4)$$

(iii) Replacing q by q^2 in (1.11) and (1.12) we get,

$${}_2\Phi_1 \left[\begin{matrix} \alpha, q^2; z \end{matrix} \right] = \frac{1}{1-} \frac{z(1-\alpha)}{\left(1 - \frac{\gamma}{q^2}\right) +} \frac{\gamma(1-q^2) \left(1 - \frac{\alpha z q^2}{\gamma}\right) / q^2}{1-} \\ \frac{z q^2(1-\alpha q^2)}{\left(1 - \frac{\gamma}{q^2}\right) +} \frac{\gamma(1-q^4) \left(1 - \frac{\alpha z q^4}{\gamma}\right) / q^2}{1-} \frac{z q^4(1-\alpha q^4)}{\left(1 - \frac{\gamma}{q^2}\right) +} \dots \quad (2.5)$$

and

$${}_3\Phi_2 \left[\begin{matrix} b, c, q^2; \frac{de}{bc} \end{matrix} \right] = \frac{1}{1-} \frac{\frac{de}{bc}(1-b)(1-e)/(1-e)(1-dq^2)}{\left(\frac{1-e/q^2}{1-e}\right) +} \\ \frac{\frac{e}{q^2}(1-q^2) \left(1 - \frac{dq^2}{b}\right) \left(1 - \frac{dq^2}{c}\right) / (1-e)(1-dq^2)(1-dq^4)}{1-} \\ \frac{\frac{deq^2}{bc}(1-dq^2)(1-bq^2)(1-cq^2)/(1-eq^2)(1-dq^4)(1-dq^6)}{\left(\frac{1-e/q^2}{1-eq^2}\right) +} \dots \quad (2.6)$$

Now, choosing $b = \alpha, c = -q^2, d = q, e = q^3$ in (1.16) and using the definition (1.6) we get

$$B(q) = \frac{1}{(1-q)^2} \lim_{\alpha \rightarrow \infty} \left\{ \frac{1}{1+} \frac{\frac{q^4}{\alpha q^2}(1-\alpha)(1+q^2)/(1-q^3)^2}{\left(\frac{1-q}{1-q^3}\right) +} \right. \\ \left. \frac{q(1-q^2) \left(1 - \frac{q^3}{\alpha}\right) (1+q)/(1-q^3)^2(1-q^5)}{1+} \right\}$$

$$\begin{aligned}
& \frac{\frac{q^4}{\alpha}(1-q^3)(1+\alpha q^2)(1+q^4)/(1-q^5)^2(1-q^7)}{\left(\frac{1-q}{1-q^5}\right) + \dots} \\
&= \frac{1}{(1-q)^2} \left\{ \frac{1}{1-} \frac{q^2(1+q^2)/(1-q^3)^2}{\left(\frac{1-q}{1-q^3}\right) +} \frac{q(1-q^2)(1+q)/(1-q^3)^2(1-q^5)}{1-} \right. \\
& \quad \left. \frac{q^6(1-q^3)(1+q^4)/(1-q^5)^2(1-q^7)}{\left(\frac{1-q}{1-q^5}\right) + \dots} \right\} \quad (2.7)
\end{aligned}$$

3. Bilateral extension of mock theta functions of order two $B(q)$

We can introduce the bilateral extension of $B(q)$ in the following form

$$B_c(q) = \sum_{n=-\infty}^{\infty} \frac{q^n [-q; q^2]_n}{[q; q^2]_{n+1}}. \quad (3.1)$$

Let us now consider the following summation formula,

$$\sum_{n=-\infty}^{\infty} \frac{[a; q]_n z^n}{[b; q]_n} = \frac{[q; q]_{\infty} [b/a; q]_{\infty} [az; q]_{\infty} [q/az; q]_{\infty}}{[b; q]_{\infty} [q/a; q]_{\infty} [z; q]_{\infty} [b/az; q]_{\infty}}. \quad (3.2)$$

$$B_c(q) = \frac{[q^4; q^4]_{\infty}}{2[q; q^2]_{\infty}^2}. \quad (3.3)$$

$$\begin{aligned}
B_c(q) &= \frac{[q^4; q^4]_{\infty}}{2[q; q^2]_{\infty}^2} = \frac{[q^4; q^4]_{\infty} [-q; q]_{\infty}^2}{2} = \frac{[q^4; q^4]_{\infty} [q^2; q^2]_{\infty}^2}{2[q; q]_{\infty}^2} \\
&= \frac{\eta(4\tau)\eta^2(2\tau)}{2q^{1/4}\eta^2(\tau)}. \quad (3.4)
\end{aligned}$$

where $\eta(\tau) = q^{1/24}[q; q]_{\infty}$, $q = e^{2\pi i\tau}$, such that $Im \tau > 0$.

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