

A Note on Some Ramanujan's Modular Equations

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Abstract: In this paper, we establish certain modular equations influenced by Srinivasa Ramanujan using Maple.

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1. Introduction

In the unorganized pages of his second notebook [11], Ramanujan recorded 23 identities involving ratio of Dedekind's eta function of which have been proved by B. C. Berndt and L. -C. Zhang [5] by employing Ramanujan's modular equations of various degree, or via his mixed modular equations or via the theory of modular forms. For the wonderful introduction to modular equations see [2], [3], [4],[10], [11], [12] and for some beautiful subsequent work on modular equations see [1], [5], [6], [7], [9], [8].

In Section 2 of this paper, we set up some preliminary results which are useful and in the final section we establish certain modular equations. We conclude this section by recalling theta functions defined by Ramanujan.

In Chapter 16 of his second notebook [11], Ramanujan develops the theory of theta function and his theta function is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}$$

$$= (-a, ab)_{\infty} (-b, ab)_{\infty} (ab, ab)_{\infty}, \quad |ab| < 1$$

where

$$(a; q)_0 := 1$$

and

$$(a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1.$$

Following Ramanujan, we define

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q, q^2)_\infty (q^2, q^2)_\infty$$

and

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2, q^2)_\infty}{(q, q^2)_\infty}.$$

2. Preliminary results

A modular equation of degree n is an equation relating α and β that is induced by

$$n \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)},$$

where

$${}_2F_1(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n, \quad |x| < 1,$$

is the ordinary or Gaussian hypergeometric function, where a , b and c are complex numbers such that c is not a nonpositive integer with

$$(a)_n := a(a+1)(a+2)\dots(a+n-1).$$

Then, we say that β is of n^{th} degree over α and call the ratio

$$m := \frac{z_1}{z_n},$$

the multiplier, where $z_1 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)$ and $z_n = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)$.

Theorem 2.1

The following are the modular equations, formulas for multipliers, and identities for degree 3:

$$(i) \quad m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}, \quad (2.1)$$

$$(ii) \quad \frac{9}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}. \quad (2.2)$$

For a proof, see [2], Entry 5 (vii), [pp. 230- 238].

Theorem 2.2 The following are the modular equations and formulas for multipliers for degree 5:

$$(i) \quad m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}, \quad (2.3)$$

$$(ii) \quad \frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}. \quad (2.4)$$

For a proof, see [2], Entry 13 (xii), [pp. 280 - 288].

Theorem 2.3 Let β and γ be of the third and ninth degrees, respectively, with respect to α . Let $m = \frac{z_1}{z_3}$ and $m' = \frac{z_3}{z_9}$. Then the following modular equations are valid:

$$(i) \quad \left(\frac{\gamma}{\alpha}\right)^{1/8} + \left(\frac{1-\gamma}{1-\alpha}\right)^{1/8} - \left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8} = \sqrt{mm'}, \quad (2.5)$$

$$(ii) \quad \left(\frac{\alpha}{\gamma}\right)^{1/8} + \left(\frac{1-\alpha}{1-\gamma}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/8} = \frac{3}{\sqrt{mm'}}. \quad (2.6)$$

For a proof, see [2], Entry 3 (x) and (xi), [pp. 352 - 358].

3. Main results

Theorem 3.1

If

$$R = \frac{\varphi^4(q)}{\varphi^4(q^3)} \quad \text{and} \quad S = \frac{\varphi^4(q^2)}{\varphi^4(q^6)}$$

then

$$\begin{aligned} & (RS)^2 + \frac{6561}{(RS)^2} + 176 \left(RS + \frac{81}{RS} \right) + \left(\frac{R}{S} \right)^2 + \left(\frac{S}{R} \right)^2 + \left[6 \left(RS + \frac{81}{RS} \right) - 1808 \right] \left(\frac{R}{S} + \frac{S}{R} \right) \\ & - 4 \left\{ \left(\sqrt[3]{RS} + \frac{729}{\sqrt[3]{RS}} \right) + 606 \left(\sqrt{RS} + \frac{9}{\sqrt{RS}} \right) \right\} \left(\left(\frac{R}{S} \right)^{1/2} + \left(\frac{S}{R} \right)^{1/2} \right) \\ & - 4 \left(\sqrt{RS} + \frac{9}{\sqrt{RS}} \right) \left(\left(\frac{R}{S} \right)^{3/2} + \left(\frac{S}{R} \right)^{3/2} \right) + 36036 = 0. \end{aligned} \quad (3.1)$$

Proof. Let

$$P = \frac{\varphi^4(-q)}{\varphi^4(-q^3)} \quad \text{and} \quad Q = \frac{\varphi^4(-q^2)}{\varphi^4(-q^6)}.$$

Using Entry 10 (ii) and (iii) of Chapter 17 [p. 122] in P and Q , it is easy to see that

$$\frac{P}{Q} = \left(\frac{1-\alpha}{1-\beta} \right)^{1/2} \quad \text{and} \quad \frac{P}{Q^2} = \left(\frac{z_3}{z_1} \right)^2.$$

Employing these in (2.1) and (2.2) respectively, it is easy to see that

$$\left(\frac{\beta}{\alpha} \right)^{1/2} = \frac{Q(Q-1)}{P-Q} \quad \text{and} \quad \left(\frac{\alpha}{\beta} \right)^{1/2} = \frac{P(Q-9)}{Q(P-Q)}.$$

Multiplying the above,

$$(P-Q)^2 = P(Q-1)(Q-9). \quad (3.2)$$

On changing q to $-q$ in (3.2),

$$(R-Q)^2 = R(Q-1)(Q-9). \quad (3.3)$$

Eliminating Q between (3.2) and (3.3)

$$P^2 R^2 + P^2 - 2P^2 R - 44PR - 2R^2 P - 18P - 18R + R^2 + 81 = 0. \quad (3.4)$$

Changing q to q^2 in (3.4), we have

$$Q^2 S^2 + Q^2 - 2Q^2 S - 44QS - 2QS^2 - 18Q - 18S + S^2 + 81 = 0. \quad (3.5)$$

Eliminating Q between (3.3) and (3.5), dividing throughout by $(RS)^2$ and on simplifying we have the result.

Theorem 3.2 If

$$R = \frac{\varphi^2(q^5)}{\varphi^2(q)} \quad \text{and} \quad S = \frac{\varphi^2(q^{10})}{\varphi^2(q^2)}$$

then

$$\begin{aligned} & 25 \left((5RS)^2 + \frac{1}{(5RS)^2} \right) - 20\sqrt{5} \left(\sqrt[3]{5RS} + \frac{1}{\sqrt[3]{5RS}} \right) \left(\sqrt{\frac{R}{S}} + \sqrt{\frac{S}{R}} \right) \\ & + 30 \left(5RS + \frac{1}{5RS} \right) \left(\frac{R}{S} + \frac{S}{R} \right) - 8\sqrt{5} \left(\sqrt{5RS} + \frac{1}{\sqrt{5RS}} \right) \left(\sqrt{\frac{R}{S}} + \sqrt{\frac{S}{R}} \right) \end{aligned}$$

$$\begin{aligned}
&= 4\sqrt{5} \left(\sqrt{5RS} + \frac{1}{\sqrt{5RS}} \right) \left(\left(\frac{R}{S} \right)^{3/2} + \left(\frac{S}{R} \right)^{3/2} \right) \\
&\quad - \left(\left(\frac{R}{S} \right)^2 + \left(\frac{S}{R} \right)^2 \right) + 224 \left(\frac{R}{S} + \frac{S}{R} \right) - 660.
\end{aligned} \tag{3.6}$$

Proof. Let

$$P = \frac{\varphi^2(-q^5)}{\varphi^2(-q)} \quad \text{and} \quad S = \frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)}.$$

Using entry 10 (ii) and (iii) of Chapter 17 [p. 122] in P and Q,

$$\frac{P}{Q} = \left(\frac{1-\beta}{1-\alpha} \right)^{1/4} \quad \text{and} \quad \frac{P}{Q^2} = \frac{z_1}{z_5}.$$

Using these in (2.3) and (2.4), we have

$$\left(\frac{\beta}{\alpha} \right)^{1/4} = \frac{P(Q-1)}{Q(P-Q)} \quad \text{and} \quad \left(\frac{\alpha}{\beta} \right)^{1/4} = \frac{Q(5Q-1)}{(P-Q)}.$$

From the above, we have

$$(P-Q)^2 = P(Q-1)(5Q-1). \tag{3.7}$$

Changing q to $-q$ in (3.7)

$$(R-Q)^2 = R(Q-1)(5Q-1). \tag{3.8}$$

Eliminating Q between (3.7) and (3.8),

$$P^2 - 10P^2R + 25P^2R^2 - 10R^2P - 4PR - 2P - 2R + R^2 + 1 = 0. \tag{3.9}$$

Replacing q by q^2 in (3.8)

$$Q^2 - 10Q^2S + 25Q^2S^2 - 10S^2Q - 4QS - 2Q - 2S + S^2 + S^2 + 1 = 0. \tag{3.10}$$

Eliminating Q between (3.8) and (3.10), dividing throughout by $(RS)^2$, and simplifying we obtain the result.

Theorem 3.3 If

$$R = \frac{\varphi(q)}{\varphi(q^9)} \quad \text{and} \quad S = \frac{\varphi(q^2)}{\varphi(q^{18})}$$

then

$$\begin{aligned}
& \left(\frac{R}{S}\right)^2 + \left(\frac{S}{R}\right)^2 - 8\left(\frac{R}{S} + \frac{S}{R}\right) + \left((RS)^2 + \frac{81}{(RS)^2}\right) + 8\left(RS + \frac{9}{RS}\right) \\
& + 6\left(RS + \frac{9}{RS}\right)\left(\frac{R}{S} + \frac{S}{R}\right) = 4\left(\sqrt{RS} + \frac{3}{\sqrt{RS}}\right)\left(\left(\frac{R}{S}\right)^{3/2} + \left(\frac{S}{R}\right)^{3/2}\right) \\
& + 4\left(\sqrt[3]{RS} + \frac{27}{\sqrt[3]{RS}}\right)\left(\left(\frac{R}{S}\right)^{1/2} + \left(\frac{S}{R}\right)^{1/2}\right) + 12. \tag{3.11}
\end{aligned}$$

Proof. Let

$$P = \frac{\varphi(-q)}{\varphi(-q^9)} \quad \text{and} \quad Q = \frac{\varphi(-q^2)}{\varphi(-q^{18})}$$

On using Entry 10 (ii) and (iii) of Chapter 17 [p. 122] in P and Q ,

$$\frac{P}{Q} = \left(\frac{1-\alpha}{1-\gamma}\right)^{1/8} \quad \text{and} \quad \frac{P}{Q^2} = \sqrt{\frac{z_9}{z_1}}.$$

Employing these in (2.5) and (2.6), we obtain

$$\left(\frac{\gamma}{\alpha}\right)^{1/8} = \frac{Q(Q-1)}{P-Q} \quad \text{and} \quad \left(\frac{\alpha}{\gamma}\right)^{1/8} = \frac{P(Q-3)}{Q(P-Q)}$$

From the above, we arrive at

$$(P-Q)^2 - P(Q-1)(Q-3) = 0. \tag{3.12}$$

Changing q to $-q$ in (3.12),

$$(R-Q)^2 - R(Q-1)(Q-3) = 0. \tag{3.13}$$

Eliminating Q between (3.12) and (3.13),

$$P^2 + P^2R^2 - 2P^2R - 2PR^2 - 6P + 4PR + R^2 - 6R + 9 = 0. \tag{3.14}$$

Replacing q by q^2 in (3.14),

$$Q^2 + Q^2S^2 - 2Q^2S - 2QS^2 - 6Q + 4QS + S^2 - 6S + 9 = 0. \tag{3.15}$$

Finally, on eliminating Q between (3.13) and (3.15), dividing throughout by $(RS)^2$ and on simplifying we have the result.

Theorem 3.4 If

$$R = \frac{\psi^4(-q)}{q\psi^4(-q^3)} \quad \text{and} \quad S = \frac{\psi^4(-q^2)}{q^2\psi^4(-q^6)}$$

then

$$\begin{aligned} & \left(\frac{R}{S}\right)^2 + \left(\frac{S}{R}\right)^2 + \frac{R}{S} + \frac{S}{R} - \left(\sqrt{RS} + \frac{9}{\sqrt{RS}}\right) \left(\left(\frac{R}{S}\right)^{3/2} + \left(\frac{S}{R}\right)^{3/2}\right) \\ &= RS + \frac{81}{RS} + 9 \left(\sqrt{RS} + \frac{9}{\sqrt{RS}}\right) \left(\left(\frac{R}{S}\right)^{1/2} + \left(\frac{S}{R}\right)^{1/2}\right) + 90. \end{aligned} \quad (3.16)$$

Proof. Let

$$P = \frac{\psi^4(q)}{q\psi^4(q^3)} \quad \text{and} \quad Q = \frac{\psi^4(q^2)}{q^2\psi^4(q^6)}.$$

On using Entry 11(i) and (iii) of Chapter 17 [p.123] in P and Q ,

$$\frac{P}{Q} = \left(\frac{\beta}{\alpha}\right)^{1/2} \quad \text{and} \quad \frac{P^2}{Q} = \left(\frac{z_1}{z_3}\right)^2.$$

Using these in (2.1) and (2.2) we obtain

$$\left(\frac{1-\beta}{1-\alpha}\right)^{1/2} = \frac{P(1-P)}{P-Q} \quad \text{and} \quad \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} = \frac{Q(9-P)}{P(P-Q)}.$$

From the above, we arrive at

$$P^2 + 8PQ + Q^2 - 9Q - P^2Q = 0. \quad (3.17)$$

Changing q to $-q$ in (3.17), we have

$$R^2 - 8RQ + Q^2 - 9Q - R^2Q = 0. \quad (3.18)$$

Eliminating Q between (3.17) and (3.18),

$$P^2 - 10PR + R^2 + P^2R - PR^2 - 9P + 9R = 0. \quad (3.19)$$

Replacing q by q^2 in (3.19), we have

$$Q^2 - 10QS + S^2 + Q^2S - QS^2 - 9Q + 9S = 0. \quad (3.20)$$

On eliminating Q between (3.18) and (3.20) and on dividing throughout by $(RS)^2$, we have the result.

Theorem 3.5 If

$$R = \frac{\psi^2(-q)}{q\psi^2(-q^5)} \quad \text{and} \quad S = \frac{\psi^2(-q^2)}{q^2\psi^2(-q^{10})}$$

then

$$\begin{aligned} RS + \frac{5}{RS} - \left(\sqrt{RS} + \frac{5}{\sqrt{RS}} \right) \left(\left(\frac{R}{S} \right)^{3/2} + \left(\frac{S}{R} \right)^{3/2} \right) \\ + 5 \left(\sqrt{RS} + \frac{5}{\sqrt{RS}} \right) \left(\left(\frac{R}{S} \right)^{1/2} + \left(\frac{S}{R} \right)^{1/2} \right) \\ = \frac{R}{S} + \frac{S}{R} + \left(\frac{R}{S} \right)^2 + \left(\frac{S}{R} \right)^2 - 30. \end{aligned} \quad (3.21)$$

Proof. Let

$$P = \frac{\psi^2(q)}{q\psi^2(q^5)} \quad \text{and} \quad Q = \frac{\psi^2(q^2)}{q^2\psi^2(q^{10})}.$$

Using Entry 11(i) and (iii) of Chapter 17 [p. 123] in P and Q ,

$$\frac{P}{Q} = \left(\frac{\beta}{\alpha} \right)^{1/2} \quad \text{and} \quad \frac{P^2}{Q} = \left(\frac{z_1}{z_3} \right)^2.$$

Employing these in (2.3) and (2.4), we find that

$$\left(\frac{1-\beta}{1-\alpha} \right)^{1/4} = \frac{P(1-P)}{P-Q} \quad \text{and} \quad \left(\frac{1-\alpha}{1-\beta} \right)^{1/4} = \frac{Q(5-P)}{P(P-Q)}.$$

From the above, we arrive at

$$P^2 + 4PQ + Q^2 - 5Q - P^2Q = 0. \quad (3.22)$$

Changing q to $-q$ in (3.22)

$$R^2 - 4RQ + Q^2 - 5Q - QR^2 = 0. \quad (3.23)$$

Eliminating Q between (3.22) and (3.23)

$$P^2 - 6PR + R^2 + P^2R - PR^2 - 5P + 5R = 0. \quad (3.24)$$

Replacing q by q^2 in (3.24),

$$Q^2 - 6QS + S^2 + Q^2S - QS^2 - 5Q + 5S = 0. \quad (3.25)$$

Eliminating Q between (3.23) and (3.25) and on dividing throughout by $(RS)^2$, we have the result.

Theorem 3.6 If

$$R = \frac{\psi(-q)}{q\psi(-q^9)} \quad \text{and} \quad S = \frac{\psi(-q^2)}{q^2\psi^2(-q^{18})}$$

then

$$\begin{aligned} & 3 \left(\sqrt{RS} + \frac{3}{\sqrt{RS}} \right) \left(\left(\frac{R}{S} \right)^{1/2} + \left(\frac{S}{R} \right)^{1/2} \right) - \left(\sqrt{RS} + \frac{3}{\sqrt{RS}} \right) \\ & \quad \times \left(\left(\frac{R}{S} \right)^{3/2} + \left(\frac{S}{R} \right)^{3/2} \right) + RS + \frac{9}{RS} \\ & = \frac{R}{S} + \frac{S}{R} + \left(\frac{R}{S} \right)^2 + \left(\frac{S}{R} \right)^2 - 12. \end{aligned} \quad (3.26)$$

Proof. Let

$$P = \frac{\psi(q)}{q\psi(q^9)} \quad \text{and} \quad Q = \frac{\psi(q^2)}{q^2\psi(q^{18})}.$$

Using Entry 11 (i) and (iii) of Chapter 17 [p. 123] in P and Q ,

$$\frac{P}{Q} = \left(\frac{\gamma}{\alpha} \right)^{1/8} \quad \text{and} \quad \frac{P^2}{Q} = \left(\frac{z_1}{z_9} \right)^{1/2}.$$

Using these in (2.5) and (2.6), it is easy to see that

$$\left(\frac{1-\gamma}{1-\alpha} \right)^{1/8} = \frac{P(1-P)}{P-Q} \quad \text{and} \quad \left(\frac{1-\alpha}{1-\gamma} \right)^{1/8} = \frac{Q(3-P)}{P(P-Q)}.$$

From the above, we arrive at

$$P^2 + 2PQ + Q^2 - 3Q - P^2Q = 0. \quad (3.27)$$

On changing q to $-q$ in (3.27)

$$R^2 - 2RQ + Q^2 - 3Q - P^2Q = 0. \quad (3.28)$$

Eliminating Q between (3.27) and (3.28), we have

$$P^2 - 4PR + R^2 + P^2R - PR^2 - 3P + 3R = 0. \quad (3.29)$$

Replacing q by q^2 in (3.29)

$$Q^2 - 4QS + S^2 + Q^2S - QS^2 - 3Q + 3S = 0. \quad (3.30)$$

Finally, on eliminating Q between (3.28) and (3.30) and on dividing throughout by $(RS)^2$, we have the result.

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