

On certain transformation formulae for q-hypergeometric series

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Abstract: In this paper, certain transformation formulae have been established by using the summation formulae for truncated q- series.

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1. Introduction, Notations and Definitions

Transformation theory is a beautiful tool of special functions that plays an important role to provide a platform for the development of the interesting and useful results. In this paper, we have established certain transformation formulae for basic hypergeometric series.

As usual, for a and q complex numbers with $|q| < 1$, define

$$[a; q]_0 = 1, \quad [a; q]_n = (1 - a)(1 - aq)(1 - aq^2) \dots (1 - aq^{n-1}), \text{ for } n \in \mathbb{N}.$$

$$[a_1, a_2, a_3, \dots, a_r; q]_n = [a_1; q]_n [a_2; q]_n \dots [a_r; q]_n$$

and

$$[a; q]_\infty = \prod_{r=0}^{\infty} (1 - aq^r).$$

An ${}_r\Phi_s$ basic hypergeometric series is define by

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r; q; z \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{[a_1, a_2, \dots, a_r; q]_n z^n}{[q, b_1, b_2, \dots, b_s; q]_n} [(-)^n q^{n(n-1)/2}]^{1+s-r}.$$

In 2010, J.M. laughlin and P. Zimmar [] established the following theorem.

If $\alpha_n(a, k)$ and $\beta_n(a, k)$ are sequences satisfying the relation

$$\beta_n(a, k) = \sum_{r=0}^n \frac{[k/a; q]_{n-r} [k; q]_{n+r}}{[q; q]_{n-r} [aq; q]_{n+r}} \alpha_r(a, k) \quad (1.1)$$

Then

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(1 - kq^{2n})[\rho_1, \rho_2; q]_n}{(1 - k)[kq/\rho_1, kq/\rho_2; q]_n} \left(\frac{aq}{\rho_1 \rho_2} \right)^n \beta_n(a, k) \\ &= \frac{[kq, kq/\rho_1 \rho_2, aq/\rho_1, aq/\rho_2; q]_{\infty}}{[kq/\rho_1, kq/\rho_2, aq/\rho_1 \rho_2, aq; q]_{\infty}} \times \\ & \times \sum_{n=0}^{\infty} \frac{[\rho_1, \rho_2; q]_n}{[aq/\rho_1, aq/\rho_2; q]_n} \left(\frac{aq}{\rho_1 \rho_2} \right)^n \alpha_n(a, k). \end{aligned} \quad (1.2)$$

If we take $k=aq$ in (1.1) and (1.2) it takes the following form,

If

$$\beta_n(a; q) = \sum_{r=0}^n \alpha_r(a, q), \quad (1.3)$$

Then

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(1 - aq^{2n+1})[\rho_1, \rho_2; q]_n}{(1 - aq)[aq^2/\rho_1, aq^2/\rho_2; q]_n} \left(\frac{aq}{\rho_1 \rho_2} \right)^n \beta_n(a, q) \\ &= \frac{(1 - aq/\rho_1)(1 - aq/\rho_2)}{(1 - aq)(1 - aq/\rho_1 \rho_2)} \times \\ & \times \sum_{n=0}^{\infty} \frac{[\rho_1, \rho_2; q]_n}{[aq/\rho_1, aq/\rho_2; q]_n} \left(\frac{aq}{\rho_1 \rho_2} \right)^n \alpha_n(a, q). \end{aligned} \quad (1.4)$$

we shall make use of (1.3) and (1.4) in order to establish our main transformation formulae.

Bailey transformation is stated as, If

$$\beta_n = \sum_{r=0}^n \alpha_r u_{n-r} v_{n+r} \quad (1.5)$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n} \quad (1.6)$$

Then under suitable convergence conditions

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \quad (1.7)$$

If we take $u_r = v_r = 1, \delta_r = z^r$ then above transformation is stated as, If

$$\beta_n = \sum_{r=0}^n \alpha_r \quad (1.8)$$

and

$$\gamma_n = \frac{z^n}{1-z} \quad (1.9)$$

Then

$$\sum_{n=0}^{\infty} \alpha_n z^n = (1-z) \sum_{n=0}^{\infty} \beta_n z^n \quad (1.10)$$

we shall also use (1.8), (1.9) and (1.10) in order to establish certain transformation and summation formulae for q -series.

In the Saalschütz's summation formula

$${}_3\Phi_2 \left[\begin{matrix} a, b, q^{-n}; q; q \\ c, abq^{1-n}/c \end{matrix} \right] = \frac{[c/a, c/b; q]_n}{[c, c/ab; q]_n} \quad (1.11)$$

[Slater 2; App. IV (IV.4)]

If we take $c=abq$ then we get,

$${}_2\Phi_1 \left[\begin{matrix} a, b; q; q \\ abq \end{matrix} \right]_n = \frac{[aq, bq; q]_n}{[q, abq; q]_n}, \quad (1.12)$$

where

$${}_2\Phi_1 \left[\begin{matrix} a, b; q; q \\ abq \end{matrix} \right]_n = \sum_{r=0}^n \frac{[a, b; q]_r q^r}{[q, abq; q]_r}, \quad (1.13)$$

is a truncated basic hypergeometric series.

In the Jackson's summation theorem

$$\begin{aligned} & {}_8\Phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, \frac{a^2 q^{n+1}}{bcd}, q^{-n}; q; q \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{bcdq^{-n}}{a}, aq^{n+1} \end{matrix} \right] \\ &= \frac{[aq, aq/bc, aq/bd, aq/cd; q]_n}{[aq/b, aq/c, aq/d, aq/bcd; q]_n} \end{aligned} \quad (1.14)$$

If we take $b=a/cd$ then we find

$${}_6\Phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a/cd, c, d; q; q \\ \sqrt{a}, -\sqrt{a}, cdq, aq/c, aq/d \end{matrix} \right]_n = \frac{[aq, dq, cq, aq/cd; q]_n}{[cdq, aq/c, aq/d, q; q]_n} \quad (1.15)$$

In the summation formula

$${}_6\Phi_5 \left[\begin{matrix} a, \sqrt{a}, -\sqrt{a}, b, c, q^{-n}; q; \frac{aq^{n+1}}{bc} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq^{1+n} \end{matrix} \right] = \frac{[aq, aq/bc; q]_n}{[aq/b, aq/c; q]_n} \quad (1.16)$$

[Slater 2; App. IV (IV.)]

if we put $c = aq^{n+1}$ then we find,

$${}_4\Phi_3 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b; q; 1/b \\ \sqrt{a}, -\sqrt{a}, aq/b \end{matrix} \right]_n = \frac{[aq, bq; q]_n}{[q, aq/b; q]_n b^n} \quad (1.17)$$

2. Transformation Formulae

In this section we shall establish transformation formulae for q-series.

(i) If we take $\alpha_r(a, q) = \frac{[a, b; q]_r q^r}{[q, abq; q]_r}$ in (1.3) then using (1.11) we get,

$$\beta_n(a, q) = \frac{[aq, bq; q]_n}{[q, abq; q]_n}. \quad (2.1)$$

Putting these values of $\alpha_n(a, q)$ and $\beta_n(a, q)$ in (1.14) we get,

$$\begin{aligned} & {}_6\Phi_5 \left[\begin{matrix} aq, q\sqrt{aq}, -q\sqrt{aq}, \rho_1, \rho_2, bq; q; \frac{aq}{\rho_1\rho_2} \\ \sqrt{aq}, -\sqrt{aq}, \frac{aq^2}{\rho_1}, \frac{aq^2}{\rho_2}, abq \end{matrix} \right] \\ &= \frac{(1 - aq/\rho_1)(1 - aq/\rho_2)}{(1 - aq)(1 - aq/\rho_1\rho_2)} {}_4\Phi_3 \left[\begin{matrix} a, \rho_1, \rho_2, b; q; \frac{aq^2}{\rho_1\rho_2} \\ \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, abq \end{matrix} \right], \end{aligned} \quad (2.2)$$

which is a transformation of a nearly well-poised series of second kind.

For $b=1$, (2.2) gives the summation formula,

$$\sum_{n=0}^{\infty} \frac{(1 - aq^{2n+1})[\rho_1, \rho_2; q]_n}{(1 - aq)[aq^2/\rho_1, aq^2/\rho_2; q]_n} = \frac{(1 - aq/\rho_1)(1 - aq/\rho_2)}{(1 - aq)(1 - aq/\rho_1\rho_2)}. \quad (2.3)$$

(ii) Again, choosing $\alpha_r(a, q) = \frac{[a, a/cd, c, d; q]_r (1 - aq^{2r})q^r}{[q, cdq, aq/c, aq/d; q]_r (1 - a)}$ in (1.3) and using (1.15) we get,

$$\beta_n(a, q) = \frac{[aq, cq, dq, aq/cd; q]_n}{[q, aq/c, aq/d, cdq; q]_n} \quad (2.4)$$

Putting these values of $\alpha_n(a, q)$ and $\beta_n(a, q)$ in (1.4) we get,

$$\begin{aligned}
 & {}_8\Phi_7 \left[\begin{matrix} aq, q\sqrt{aq}, -q\sqrt{aq}, \rho_1, \rho_2, cq, dq, aq/cd; q; \frac{aq}{\rho_1\rho_2} \\ \sqrt{aq}, -\sqrt{aq}, \frac{aq^2}{\rho_1}, \frac{aq^2}{\rho_2}, \frac{aq}{c}, \frac{aq}{d}, cdq \end{matrix} \right] \\
 &= \frac{(1 - aq/\rho_1)(1 - aq/\rho_2)}{(1 - aq)(1 - aq/\rho_1\rho_2)} \times \\
 & {}_8\Phi_7 \left[\begin{matrix} q, q\sqrt{a}, -q\sqrt{a}, \rho_1, \rho_2, c, d, a/cd; q; \frac{aq^2}{\rho_1\rho_2} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{aq}{c}, \frac{aq}{d}, cdq \end{matrix} \right]. \tag{2.5}
 \end{aligned}$$

(iii) Choosing $\alpha_r(a, q) = \frac{[a, q\sqrt{a}, -q\sqrt{a}, b; q]_r}{[q, \sqrt{a}, -\sqrt{a}, aq/b; q]_r} \frac{1}{b^r}$ in (1.3) then using (1.17) we get,

$$\beta_n(a, q) = \frac{[aq, bq; q]_n}{[q, aq/b; q]_n b^n}.$$

Putting these values of $\alpha_n(a, q)$ and $\beta_n(a, q)$ in (1.4) we have,

$$\begin{aligned}
 & {}_6\Phi_5 \left[\begin{matrix} aq, q\sqrt{aq}, -q\sqrt{aq}, \rho_1, \rho_2, bq; q; \frac{aq}{\rho_1\rho_2b} \\ \sqrt{aq}, -\sqrt{aq}, \frac{aq^2}{\rho_1}, \frac{aq^2}{\rho_2}, \frac{aq}{b} \end{matrix} \right] \\
 &= \frac{(1 - aq/\rho_1)(1 - aq/\rho_2)}{(1 - aq)(1 - aq/\rho_1\rho_2)} \times \\
 & {}_6\Phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \rho_1, \rho_2, b; q; \frac{aq}{b\rho_1\rho_2} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2}, \frac{aq}{b} \end{matrix} \right]. \tag{2.6}
 \end{aligned}$$

3. Transformation formulae for truncated basic hypergeometric series

In this section we shall deduce certain transformation formulae for truncated q -series by making use known transformations. The results established in this section will be used in next section.

(i) Sear's ${}_4\Phi_3$ transformation formulae is

$${}_4\Phi_3 \left[\begin{matrix} a, b, c, q^{-n}; q; q \\ d, e, \frac{abcq^{1-n}}{de} \end{matrix} \right] = \frac{\left[\frac{e}{a}, \frac{bcq^{1-n}}{de}; q \right]_n a^n}{\left[e, \frac{abcq^{1-n}}{de}; q \right]_n} \times$$

$$\times {}_4\Phi_3 \left[\begin{matrix} a, d/b, d/c, q^{-n}; q; q \\ d, aq^{1-n}/e, \frac{de}{bc} \end{matrix} \right] \quad (3.1)$$

[Gasper and Rahman; (2.10.4) p. 49]

If we take $e = q^{-n}$ in (3.1) we get,

$${}_3\Phi_2 \left[\begin{matrix} a, b, c; q; q \\ d, \frac{abcq}{d} \end{matrix} \right]_n = \frac{\left[aq, \frac{bcq}{d}; q \right]_n}{\left[q, \frac{abcq}{d}; q \right]_n} {}_4\Phi_3 \left[\begin{matrix} a, d/b, d/c, q^{-n}; q; q \\ d, aq, \frac{dq^{-n}}{bc} \end{matrix} \right] \quad (3.2)$$

(ii) Let us consider the following transformation

$$\begin{aligned} & {}_8\Phi_7 \left[\begin{matrix} q, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-n}; q; \frac{a^2q^{n+2}}{bcde} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, aq^{1+n} \end{matrix} \right] \\ &= \frac{[aq, \lambda q/e; q]_n}{[\lambda q, aq/e; q]_n} \\ &\times {}_8\Phi_7 \left[\begin{matrix} \lambda, q\sqrt{\lambda}, -q\sqrt{\lambda}, \lambda b/a, \lambda c/a, \lambda d/a, e, q^{-n}; q; \frac{aq^{n+1}}{e} \\ \sqrt{\lambda}, -\sqrt{\lambda}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{\lambda q}{e}, \lambda q^{n+1} \end{matrix} \right], \end{aligned} \quad (3.3)$$

where $\lambda = a^2q/bcd$.

[Gasper and Rahman; (2.10.3) p. 49]

Putting $e = aq^{n+1}$ in (3.3) we find,

$$\begin{aligned} & {}_6\Phi_5 \left[\begin{matrix} q, q\sqrt{a}, -q\sqrt{a}, b, c, d; q; \frac{q}{bcd} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d} \end{matrix} \right]_n \\ &= \frac{[aq, aq/\lambda; q]_n (\lambda/a)^n}{[q, \lambda q; q]_n} \\ &\times {}_8\Phi_7 \left[\begin{matrix} \lambda, q\sqrt{\lambda}, -q\sqrt{\lambda}, \lambda b/a, \lambda c/a, \lambda d/a, e, q^{-n}; q; 1 \\ \sqrt{\lambda}, -\sqrt{\lambda}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{\lambda q^{-n}}{e}, \lambda q^{n+1} \end{matrix} \right], \end{aligned} \quad (3.4)$$

where $\lambda = a^2q/bcd$.

(iii) Let us consider the Watson's transformation,

$${}_8\Phi_7 \left[\begin{matrix} q, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-n}; q; \frac{a^2q^{n+2}}{bcde} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, aq^{1+n} \end{matrix} \right]$$

$$= \frac{[aq, aq/de; q]_n}{[aq/d, aq/e; q]_n} {}_4\Phi_3 \left[\begin{matrix} q^{-n}, d, e, aq/bc; q; q \\ aq/b, aq/c, deq^{-n}/a \end{matrix} \right]. \quad (3.5)$$

Putting $e = aq^{n+1}$ in (3.5) we find,

$$\begin{aligned} & {}_6\Phi_5 \left[\begin{matrix} q, q\sqrt{a}, -q\sqrt{a}, b, c, d, e; q; \frac{aq}{bcd} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d} \end{matrix} \right]_n \\ &= \frac{[aq, dq; q]_n}{[q, aq/d; q]_n d^n} {}_4\Phi_3 \left[\begin{matrix} q^{-n}, d, aq^{n+1}, aq/bc; q; q \\ aq/b, aq/c, dq \end{matrix} \right]. \end{aligned} \quad (3.6)$$

4. Double series transformation formulae

In this section we shall establish certain transformation of double series by using the results of section 3.

(i) If we take $\alpha_r(a, q) = \frac{[a, b, c; q]_r q^r}{[q, d, abcq/d; q]_r}$ in (1.3) then using (3.2) we find,

$$\beta_n(a, q) = \frac{[aq, bcq/d; q]_n}{[q, abcq/d; q]_n} {}_4\Phi_3 \left[\begin{matrix} q^{-n}, a, d/b, d/c; q; q \\ d, aq, dq^{-n}/bc \end{matrix} \right]. \quad (4.1)$$

Putting these values of $\alpha_n(a, q)$ and $\beta_n(a, q)$ in (1.4) we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(1 - aq^{2n+1})[\rho_1, \rho_2; q]_n [aq, bcq/d; q]_n}{(1 - aq)[aq^2/\rho_1, aq^2/\rho_2; q]_n [q, abcq/d; q]_n} \left(\frac{aq}{\rho_1 \rho_2} \right)^n \times \\ & {}_4\Phi_3 \left[\begin{matrix} q^{-n}, a, d/b, d/c; q; q \\ d, aq, dq^{-n}bc \end{matrix} \right] \\ &= \frac{(1 - aq/\rho_1)(1 - aq/\rho_2)}{(1 - aq)(1 - aq/\rho_1 \rho_2)} {}_5\Phi_4 \left[\begin{matrix} a, b, c, \rho_1, \rho_2; q; \frac{aq^2}{\rho_1 \rho_2} \\ d, \frac{abcq}{d}, \frac{aq}{\rho_1}, \frac{aq}{\rho_2} \end{matrix} \right] \end{aligned} \quad (4.2)$$

Taking $d=c$ in (4.2) we have

$$\begin{aligned} & {}_6\Phi_5 \left[\begin{matrix} aq, q\sqrt{aq}, -q\sqrt{aq}, \rho_1, \rho_2, bq; q; \frac{aq}{\rho_1 \rho_2} \\ \sqrt{aq}, -\sqrt{aq}, \frac{aq^2}{\rho_1}, \frac{aq^2}{\rho_2}, abq \end{matrix} \right] \\ &= \frac{(1 - aq/\rho_1)(1 - aq/\rho_2)}{(1 - aq)(1 - aq/\rho_1 \rho_2)} {}_4\Phi_3 \left[\begin{matrix} a, b, \rho_1, \rho_2; q; \frac{aq^2}{\rho_1 \rho_2} \\ abq, \frac{aq}{\rho_1}, \frac{aq}{\rho_2} \end{matrix} \right] \end{aligned} \quad (4.3)$$

which is precisely (2.2)

Taking $d=bc$ in (4.2) we get,

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\frac{1 - aq^{2n+1}}{1 - aq} \right) \frac{[\rho_1, \rho_2; q]_n}{\left[q, \frac{aq^2}{\rho_1}, \frac{aq^2}{\rho_2}; q \right]_n} {}_3\Phi_2 \left[\begin{matrix} a, b, c; q; q \\ bc, aq \end{matrix} \right]_n \\ &= \frac{(1 - aq/\rho_1)(1 - aq/\rho_2)}{(1 - aq)(1 - aq/\rho_1\rho_2)} {}_5\Phi_4 \left[\begin{matrix} a, b, c, \rho_1, \rho_2; q; \frac{aq^2}{\rho_1\rho_2} \\ bc, aq, \frac{aq}{\rho_1}, \frac{aq}{\rho_2} \end{matrix} \right] \end{aligned} \quad (4.4)$$

(ii) Again, if we take $\alpha_r(a, q) = \frac{[a, b, c; q]_r q^r}{[q, d, abcq/d; q]_r}$ in (1.8) then using (3.2) we have

$$\beta_n(a, q) = \frac{[aq, bcq/d; q]_n}{[q, abcq/d; q]_n} {}_4\Phi_3 \left[\begin{matrix} q^{-n}, a, d/b, d/c; q; q \\ d, aq, dq^{-n}/bc \end{matrix} \right].$$

Putting these values in (1.10) we get,

$$\begin{aligned} & {}_3\Phi_2 \left[\begin{matrix} a, b, c; q; zq \\ d, abcq/d \end{matrix} \right] \\ &= (1 - z) \sum_{n=0}^{\infty} \frac{[aq, bcq/d; q]_n z^n}{[q, abcq/d; q]_n} {}_4\Phi_3 \left[\begin{matrix} q^{-n}, a, d/b, d/c; q; q \\ d, aq, dq^{-n}/bc \end{matrix} \right]. \end{aligned} \quad (4.5)$$

If we take $d=bc$ in (4.5) then we get,

$${}_3\Phi_2 \left[\begin{matrix} a, b, c; q; zq \\ bc, aq \end{matrix} \right] = (1 - z) \sum_{n=0}^{\infty} z^n {}_3\Phi_2 \left[\begin{matrix} a, b, c; q; q \\ aq, bc \end{matrix} \right]_n \quad (4.6)$$

A number of similar results can also be established.

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