

On certain transformation formulae involving basic hypergeometric functions

Bindu Prakash Mishra,
 Department of Mathematics,
 M.D. College, Parel, Mumbai-400012, India

Abstract: In this paper, we establish certain transformations for unilateral as well as bilateral basic hypergeometric functions.

Key words and phrases: Transformation formulae, unilateral basic hypergeometric series, q-series, bilateral q-series, basic hypergeometric functions.

2000 A.M.S. subject classification: 33A30

1. Introduction, Notations and Definitions

Somashekara and Fathima [2], in order to establish certain continued fraction representation for the ratio of two ${}_2\Psi_2$, established certain functional relations satisfied by ${}_2\Psi_2$, with the help of certain simple identities. In this paper we discuss how these identities can be utilized to establish interesting transformations involving basic hypergeometric functions which may prove useful in the study of transformation theory of q-series.

$$[\alpha; q]_n = [\alpha]_n = \begin{cases} (1 - \alpha)(1 - \alpha q)(1 - \alpha q^2) \dots (1 - \alpha q^{n-1}); & n > 0 \\ 1; & n = 0 \end{cases}$$

$$[\alpha; q]_\infty = \prod_{r=0}^{\infty} (1 - \alpha q^r) \quad \text{and} \quad [\alpha; q]_{-n} = \frac{(-)^n q^{n(n+1)/2}}{\alpha^n [q/\alpha; q]_n}$$

Following the above notation, we define

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r; q; z \end{matrix} \middle| \begin{matrix} b_1, b_2, \dots, b_s \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{[a_1, a_2, \dots, a_r; q]_n z^n}{[q, b_1, b_2, \dots, b_s]_n}, \quad (1.1)$$

max. $(|q|, |z| < 1)$. where

$$[a_2, a_2, \dots, a_r; q]_n = [a_1; q]_n [a_2; q]_n \dots [a_r; q]_n.$$

Also, we define the basic bilateral hypergeometric function

$${}_r\Psi_r \left[\begin{matrix} a_1, a_2, \dots, a_r; q; z \end{matrix} \middle| \begin{matrix} b_1, b_2, \dots, b_r \end{matrix} \right] = \sum_{n=-\infty}^{\infty} \frac{[a_1, a_2, \dots, a_r; q]_n z^n}{[b_1, b_2, \dots, b_r]_n} \quad (1.2)$$

where, for convergence $|b_1 b_2 \dots b_r / a_1 a_2 \dots a_r| < z < 1$.

We shall make use of the following transformations and summations,

$${}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c, d \end{matrix} \right] = \frac{[az, d/a, c/b, dq/abz; q]_\infty}{[z, d, q/b, cd/abz; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abz/d; q; d/a \\ az, c \end{matrix} \right] \quad (1.3)$$

[Gasper-Rahman 1; (5.20 (i) p. 137)]

$${}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c, d \end{matrix} \right] = \frac{[az, bz, cq/abz, dq/abz; q]_\infty}{[q/a, q/b, c, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abz/c, abz/d; q; cd/abz \\ az, bz \end{matrix} \right] \quad (1.4)$$

[Gasper-Rahman 1; (5.20 (ii) p. 137)]

$${}_2\Psi_2 \left[\begin{matrix} a, b; q; -\alpha q/ab \\ \alpha q/a, \alpha q/b \end{matrix} \right] = \frac{[\alpha q/ab; q]_\infty [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_\infty}{[\alpha q/a, \alpha q/b, q/a, q/b, -\alpha q/ab; q]_\infty} \quad (1.5)$$

[Gasper-Rahman 1; App. II (II.30) p. 239]

$${}_2\Phi_1 \left[\begin{matrix} a, b; q; z \\ c \end{matrix} \right] = \frac{[b, az; q]_\infty}{[c, z; q]_\infty} {}_2\Phi_1 \left[\begin{matrix} z, c/b; q; b \\ az \end{matrix} \right] \quad (1.6)$$

[Gasper-Rahman 1; App. III (III.1)]

$${}_2\Phi_1 \left[\begin{matrix} a, b; q; z \\ c \end{matrix} \right] = \frac{[c/b, bz; q]_\infty}{[c, z; q]_\infty} {}_2\Phi_1 \left[\begin{matrix} abz/c, b; q; c/b \\ bz \end{matrix} \right] \quad (1.7)$$

[Gasper-Rahman 1; App. III (III.2)]

$${}_2\Phi_1 \left[\begin{matrix} a, b; q; z \\ c \end{matrix} \right] = \frac{[abz/c; q]_\infty}{[z; q]_\infty} {}_2\Phi_1 \left[\begin{matrix} c/a, c/b; q; abz/c \\ c \end{matrix} \right] \quad (1.8)$$

[Gasper-Rahman 1; App. III (III.3)]

$${}_3\Psi_3 \left[\begin{matrix} a, b, d; q; q^2/abd \\ q^2/a, q^2/b, q^2/d \end{matrix} \right] = \frac{[q, q^2/ab, q^2/ad, q^2/bd; q]_\infty}{[q^2/a, q^2/b, q^2/d, q^2/abd; q]_\infty} \quad (1.9)$$

[Gasper-Rahman 1; App. II (II.30) p. 239]

$${}_3\Psi_3 \left[\begin{matrix} a, b, d; q; q^2/abd \\ q^2/a, q^2/b, q^2/d \end{matrix} \right] = \frac{[q, q^2/ab, q^2/ad, q^2/bd; q]_\infty}{[q^2/a, q^2/b, q^2/d, q^2/abd; q]_\infty} \quad (1.10)$$

[Gasper-Rahman 1; 5.18(ii) p. 137]

2. Main Results

In this section we shall establish our transformation formulae. To do so let us consider the following simple identity,

$$\frac{a(1-c)}{[c; q]_n} - \frac{(a-c)}{[cq; q]_n} = \frac{c(1-aq^n)}{[cq; q]_n} \quad (2.1)$$

Multiplying both sides of (2.1) by $\frac{[a; q]_n [b; q]_n z^n}{[d; q]_n}$ and then summing over n from $-\infty$ to ∞ , we get

$$a(1-c) {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c, d \end{matrix} \right] = (a-c) {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ cq, d \end{matrix} \right] + c(1-a) {}_2\Psi_2 \left[\begin{matrix} aq, b; q; z \\ cq, d \end{matrix} \right] \quad (2.2)$$

We shall make use of (2.2) in order to establish transformation formulae.

Transforming each ${}_2\Psi_2$ on the right side of (2.2) with the help of (1.3), we get

$$\begin{aligned} a(1-c) {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c, d \end{matrix} \right] &= (a-c) \frac{[az, d/a, cq/b, dq/abz; q]_\infty}{[z, d, q/b, cdq/abz; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abz/d; q; d/a \\ az, cq \end{matrix} \right] \\ &+ c(1-a) \frac{[azq, d/aq, cq/b, d/abz; q]_\infty}{[z, d, q/b, cd/abz; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} aq, abzq/d; q; d/aq \\ azq, cq \end{matrix} \right] \end{aligned} \quad (2.3)$$

Now, taking $c = \alpha q/a$, $d = \alpha q/b$ and $z = -\alpha q/ab$ in (2.3) and summing the ${}_2\Psi_2$ on the left of (2.3) with the help of (1.5), we get

$$\begin{aligned} (a - \alpha q/a) \frac{[\alpha q/ab, -\alpha q/b, -q/b; q]_\infty}{[-\alpha q^2/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q/ab \\ -\alpha q/b, \alpha q^2/a \end{matrix} \right] \\ = a(1 - \alpha q/ab) \frac{[\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_\infty}{[\alpha q/a, q/a; q]_\infty} \\ - \frac{\alpha q(1-a)[- \alpha q^2/b, \alpha/ab, -1/b; q]_\infty}{a[- \alpha q/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} aq, -bq; q; \alpha/ab \\ \alpha q^2/a, \alpha q^2/b \end{matrix} \right] \end{aligned} \quad (2.4)$$

Now, transforming the first ${}_2\Psi_2$ on the right of (2.2) with the help of (1.3) and the second ${}_2\Psi_2$ with the help of (1.4), we get

$$\begin{aligned} a(1-c) {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c, d \end{matrix} \right] &= (a-c) \frac{[az, d/a, cq/b, dq/abz; q]_\infty}{[z, d, q/b, cdq/abz; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abz/d; q; d/a \\ az, cq \end{matrix} \right] \\ &+ c(1-a) \frac{[azq, bz, cq/abz, d/abz; q]_\infty}{[1/a, q/b, cq, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abz/c, abzq/d; q; cd/abz \\ azq, bz \end{matrix} \right] \end{aligned} \quad (2.5)$$

Now ,taking $c = \alpha q/a, d = \alpha q/b$ and $z = \alpha q/ab$ in (2.5) and summing the ${}_2\Psi_2$ on its left with the help of (1.5), we get the following transformation

$$\begin{aligned}
& (a - \alpha q/a) \frac{[-\alpha q/b, \alpha q/ab, -q/b; q]_\infty}{[-\alpha q/ab, -\alpha q^2/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q/ab \\ -\alpha q/ab, \alpha q^2/a \end{matrix} \right] \\
&= \frac{a[\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_\infty}{[\alpha q^2/a, -\alpha q/ab, q/a; q]_\infty} \\
&+ \alpha q(1 - 1/a) \frac{[-\alpha q^2/b, -\alpha q/a, -q/a, -1/b; q]_\infty}{[\alpha q/b, 1/a, \alpha q^2/a; q]_\infty} \\
&\quad \times {}_2\Psi_2 \left[\begin{matrix} -a, -bq; q; -\alpha q/ab \\ -\alpha q^2/b, -\alpha q/a \end{matrix} \right] \tag{2.6}
\end{aligned}$$

Again transforming each ${}_2\Psi_2$ functions on the right of (2.2) with the help of (1.4),we get

$$\begin{aligned}
& a(1 - c) {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c, d \end{matrix} \right] \\
&= (a - c) \frac{[azq, bz, cq^2/abz, dq/abz; q]_\infty}{[q/a, q/b, cq, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abz/cq, abz/d; q; cdq/abz \\ az, bz \end{matrix} \right] \\
&+ c(1 - a) \frac{[azq, bz, cq/abz, d/abz; q]_\infty}{[1/a, q/b, cq, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abz/c, abzq/d; q; cd/abz \\ azq, bz \end{matrix} \right] \tag{2.7}
\end{aligned}$$

Now, taking $c = \alpha q/a, d = \alpha q/b$ and $z = \alpha q/ab$ in (2.7) and summing the ${}_2\Psi_2$ on its left with the help of (1.5),we get

$$\begin{aligned}
& a(1 - \alpha q/a^2) [-\alpha q/a, -\alpha q/b, -q^2/a, -q/b; q]_\infty \left[\begin{matrix} -a/q, -b; q; -\alpha q^2/ab \\ -\alpha q/a, -\alpha q/b \end{matrix} \right] \\
&= \frac{a[\alpha q/ab; q]_\infty [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_\infty}{[-\alpha q/ab; q]_\infty} \\
&+ \alpha q [-\alpha q^2/b, -\alpha q/a, -q/a, -1/b; q]_\infty {}_2\Psi_2 \left[\begin{matrix} -a, -bq; q; -\alpha q/ab \\ -\alpha q^2/b, -\alpha q/a \end{matrix} \right] \tag{2.8}
\end{aligned}$$

Multiplying both sides of the identity (2.1) by $[a, b; q]_n z^n / [q; q]_n$ and summing over n from 0 to ∞ , we get

$$\begin{aligned}
& a(1 - c) {}_2\Phi_1 \left[\begin{matrix} a, b; q; z \\ c \end{matrix} \right] \\
&= (a - c) {}_2\Phi_1 \left[\begin{matrix} a, b; q; z \\ cq \end{matrix} \right] + c(1 - a) {}_2\Phi_1 \left[\begin{matrix} aq, bq; q; z \\ cq \end{matrix} \right] \tag{2.9}
\end{aligned}$$

Transforming each of the series on the right of (2.9) with the help of (1.8), we get

$$\begin{aligned} a(1-c) {}_2\Phi_1 \left[\begin{matrix} a, b; q; z \\ c \end{matrix} \right] &= (a-c) \frac{[abz/cq; q]_\infty}{[z; q]_\infty} {}_2\Phi_1 \left[\begin{matrix} cq/a, cq/b; q; abz/cq \\ cq \end{matrix} \right] \\ &+ c(1-a) \frac{[abz/c; q]_\infty}{[z; q]_\infty} {}_2\Phi_1 \left[\begin{matrix} c/a, cq/b; q; abz/c \\ cq \end{matrix} \right] \end{aligned} \quad (2.10)$$

Taking $b = qn$ and $z = q$ in (2.10) and summing the ${}_2\Phi_1$ on the left of it with the help of a known result (cf. Gasper-Rahman [1; App.II(II.6)]), we get

$$\begin{aligned} (a-c)[aq^{-n}/c; q]_\infty {}_2\Phi_1 \left[\begin{matrix} cq/a, cq^{1+n}; q; aq^{-n}/c \\ cq \end{matrix} \right] &= \frac{[q, c/a; q]_n a^{n+1}}{[cq; q]_{n-1}} \\ &- c(1-a)[aq^{1-n}; q]_\infty {}_2\Phi_1 \left[\begin{matrix} c/a, cq^{1+n}; q; aq^{1-n}/c \\ cq \end{matrix} \right] \end{aligned} \quad (2.11)$$

Transforming each ${}_2\Phi_1$ on the right of (2.9) with the help of (1.7), we get

$$\begin{aligned} a(1-c) {}_2\Phi_1 \left[\begin{matrix} a, b; q; z \\ c \end{matrix} \right] &= (a-c) \frac{[cq/b, bz; q]_\infty}{[cq, z; q]_\infty} {}_2\Phi_1 \left[\begin{matrix} abz/cq, b; q; cq/b \\ bz \end{matrix} \right] \\ &+ c(1-a) \frac{[cq/b, bz; q]_\infty}{[cq, z; q]_\infty} {}_2\Phi_1 \left[\begin{matrix} abz/c, b; q; cq/b \\ bz \end{matrix} \right] \end{aligned} \quad (2.12)$$

Now, setting $a = q^{-n}$ and $z = q$ in (2.12) and summing the ${}_2\Phi_1$ on its left, we get

$$\begin{aligned} (1-cq^n) {}_2\Phi_1 \left[\begin{matrix} bq^{-n}/c, b; q; cq/b \\ bq \end{matrix} \right] &= (1-c)b^n \frac{[c/b, q, cq; q]_\infty}{[c, bq, cq/b; q]_\infty} \\ &+ c(1-q^n) {}_2\Phi_1 \left[\begin{matrix} bq^{1-n}/c, b; q; cq/b \\ bq \end{matrix} \right] \end{aligned} \quad (2.13)$$

Now, transforming ${}_2\Phi_1 \left[\begin{matrix} a, b; q; z \\ cq \end{matrix} \right]$ on the right of (2.9) with the help of (1.8) and ${}_2\Phi_1 \left[\begin{matrix} aq, b; q; z \\ cq \end{matrix} \right]$ with the help of (1.7), we get

$$\begin{aligned} a(1-c) {}_2\Phi_1 \left[\begin{matrix} a, b; q; z \\ c \end{matrix} \right] &= (a-c) \frac{[abz/cq; q]_\infty}{[z; q]_\infty} {}_2\Phi_1 \left[\begin{matrix} cq/a, cq/b; q; abz/cq \\ cq \end{matrix} \right] \\ &+ c(1-a) \frac{[cq/b, bz; q]_\infty}{[cq, z; q]_\infty} {}_2\Phi_1 \left[\begin{matrix} abz/c, b; q; cq/b \\ bz \end{matrix} \right] \end{aligned} \quad (2.14)$$

Taking $a = q^{-n}$ and $z = q$ in (2.14) and summing ${}_2\Phi_1$ on its left, we get

$$(1 - cq^n)[bq^{-n}/c; q]_\infty {}_2\Phi_1 \left[\begin{matrix} cq^{1+n}, cq/b; q; bq^{-n}/c \end{matrix} \right] = \frac{b^n[q; q]_\infty [c/b; q]_n}{[cq; q]_{n-1}} \\ + c(1 - q^n) \frac{[bq, cq/b; q]_\infty}{[cq; q]_\infty} {}_2\Phi_1 \left[\begin{matrix} bq^{1-n}/c, b; q; cq/b \end{matrix} \right] \quad (2.15)$$

Multiplying both sides of (2.1) by $\frac{[a, b, d; q]_n z^n}{[e, f; q]_n}$ and summing it over n from $-\infty$ to ∞ , we get

$$a(1 - c) {}_3\Psi_3 \left[\begin{matrix} a, b, d; q; z \\ c, e, f \end{matrix} \right] = (a - c) {}_3\Psi_3 \left[\begin{matrix} a, b, d; q; z \\ cq, e, f \end{matrix} \right] \\ + c(1 - a) {}_3\Psi_3 \left[\begin{matrix} aq, b, d; q; z \\ cq, e, f \end{matrix} \right] \quad (2.16)$$

Taking $c = q^2/a, e = q/b, f = q/d$ and $z = q/abd$ in (2.16) and summing the ${}_3\Psi_3$ on its left with the help of a known result (cf. Gasper-Rahman [1; App.II.II 31]), we get

$$(a - q/a) {}_3\Psi_3 \left[\begin{matrix} a, b, d; q; q/abd \\ q^2/a, q/b, q/d \end{matrix} \right] = q(1 - q/a) \frac{[q, q/ab, q/ad, q/bd; q]_\infty}{[q/a, q/b, q/d, q/abd; q]_\infty} \\ + q(1 - 1/a) {}_3\Psi_3 \left[\begin{matrix} aq, b, d; q; q/abd \\ q^2/a, q/b, q/d \end{matrix} \right] \quad (2.17)$$

Finally, taking $c = q^2/a, e = q^2/b, f = q^2/d$ and $z = q^2/abd$ in (2.16) and summing the ${}_3\Psi_3$ on its left by a known result (cf. Gasper-Rahman [1; 5.18(ii), p.137], we get

$$a(1 - q^2/a^2) {}_3\Psi_3 \left[\begin{matrix} a, b, d; q; q^2/abd \\ q^3/a, q^2/b, q^2/d \end{matrix} \right] = a(1 - q^2/a) \frac{[q, q^2/ab, q^2/ad, q^2/bd; q]_\infty}{[q^2/a, q^2/b, q^2/d, q^2/abd; q]_\infty} \\ + q^2(1 - 1/a) {}_3\Psi_3 \left[\begin{matrix} aq, b, d; q; q^2/abd \\ q^3/a, q^2/b, q^2/d \end{matrix} \right] \quad (2.18)$$

Similarly, several other transformations can also be established. The transformations investigated here may be of use in the study of mock theta functions.

Acknowledgement

The author is thankful to University Grants Commission, New Delhi for sanctioning major research project no. F.38-122/2009 (SR) dated 19-12-2009 and also thankful to Dr. S.N. Singh, Department of Mathematics, T.D.P.G. College, Jaunpur, for his valuable guidance in the preparation of this paper.

References

- [1] Gasper, G. and Rahman, M.: Basic hypergeometric series, Cambridge University Press (1991).
- [2] Somashekara, D. D. and Fathima, S. N.: On continued fraction expansions for the ratio of ${}_2\Psi_2$, Indian Journal of Mathematics, Vol.45, No.3(2003), 333-355.