

On Ramanujan's Theta Functions

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Abstract: In this paper we shall attempt to establish certain Lambert series representations for Ramanujan's theta functions and evaluate certain integrals involving these theta functions with the help of Lambert series representations.

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1. Introduction, Notations and Definitions

Ramanujan's general theta function $f(a, b)$ is defined by

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1 \quad (1.1)$$

Most important special cases of (1.1) are given by

$$\begin{aligned} \Phi(q) = f(q, q) &= \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{[-q; -q]_{\infty}}{[q; -q]_{\infty}} = \frac{[q^2; q^2]_{\infty} [-q; q^2]_{\infty}}{[q; q^2]_{\infty} [-q^2; q^2]_{\infty}} \\ &= [q^2; q^2]_{\infty} [-q; q^2]_{\infty}, \end{aligned} \quad (1.2)$$

$$\Psi(q) = f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{[q^2; q^2]_{\infty}}{[-q; q^2]_{\infty}} \quad (1.3)$$

$$f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-)^n q^{n(3n+1)/2} = [q; q]_{\infty}. \quad (1.4)$$

we shall also have the occasion to use the function $\chi(q)$ given by

$$\chi(-q) = [q; q^2]_{\infty}, \quad (1.5)$$

where

$$[\alpha; p]_{\infty} = \prod_{n=0}^{\infty} (1 - \alpha p^n).$$

Borwin discovered cubic theta functions defined by

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{n^2+mn+n^2} \quad (1.6)$$

$$b(q) = \sum_{m,n=-\infty}^{\infty} q^{n^2+mn+m^2} \omega^{n-m}, \quad (\omega = e^{2\pi i/3}) \quad (1.7)$$

and

$$c(q) = q^{1/3} \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n} \quad (1.8)$$

[1; Chapter 33; (2.2)-(2.4), pp.93]

We also have

$$b(q) = \frac{f^3(-q)}{f(-q^3)} = \frac{[q; q^3]_{\infty}^3}{[q^3; q^3]_{\infty}} \quad (1.9)$$

$$c(q) = 3q^{1/3} \frac{f^3(-q^3)}{f(-q)} = 3q^{1/3} \frac{[q^3; q^3]_{\infty}^3}{[q; q]_{\infty}} \quad (1.10)$$

[1; Chapter 33; (5.4)-(5.5), pp.109]

Borwin establish the following representation for $a(q)$ in terms of Lambert series

$$a(q) = 1 + 6 \sum_{n=0}^{\infty} \left(\frac{q^{3n+1}}{1 - q^{3n+1}} - \frac{q^{3n+2}}{1 - q^{3n+2}} \right) \quad (1.11)$$

[1; Chapter 33; (2.6), pp.93]

We shall use the following two known results,

$$\sum_{n=-\infty}^{\infty} \frac{[\alpha; q]_n z^n}{[\beta; q]_n} = \frac{[q, \beta/\alpha, \alpha z, q/\alpha z; q]_{\infty}}{[\beta, q/\alpha, z, \beta/\alpha z]_{\infty}} \quad (1.12)$$

[Ramanujan 3; p. 196 entry 17]

and

$$\sum_{k=0}^{\infty} \frac{(k+1)[q/\alpha; q]_k \alpha^k}{[\beta; q]_{k+1}} + \sum_{k=0}^{\infty} \frac{k[q/\beta; q]_k \beta^k}{[\alpha; q]_{k+1}} = \frac{[q; q]_{\infty}^3 [\alpha\beta; q]_{\infty}}{[\alpha; q]_{\infty}^2 [\beta; q]_{\infty}^2} \quad (1.13)$$

[Bhargava 2; (2.3) p. 556]

2. Main Results

In this section we shall establish our results. If we substitute $\beta = \alpha q$ in (1.12), we get

$$\sum_{n=0}^{\infty} \frac{z^n}{1 - \alpha q^n} - \frac{q}{\alpha z} \sum_{n=0}^{\infty} \frac{(q/z)^n}{1 - q^{n+1}/\alpha} = \frac{[q, q, \alpha z, q/\alpha z; q]_{\infty}}{[\alpha, q/\alpha, z, q/z; q]_{\infty}} \quad (2.1)$$

Next, setting $\beta = q/\alpha$ in (1.13), we get,

$$\sum_{k=0}^{\infty} \frac{(k+1)\alpha^k}{1 - q^{k+1}/\alpha} + \sum_{k=0}^{\infty} \frac{k(q/\alpha)^k}{1 - \alpha q^k} = \frac{[q; q]_{\infty}^4}{[\alpha; q]_{\infty}^2 [q/\alpha; q]_{\infty}^2} \quad (2.2)$$

Now, replacing q by q^4 and then setting $\alpha = z = q$ in (2.1), we get

$$\Psi^2(q) = \sum_{n=0}^{\infty} \frac{q^n}{1 - q^{4n+1}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1 - q^{4n+3}} \quad (2.3)$$

Again, replacing q by q^4 and then setting $\alpha = -q$ and $z=q$ in (2.1), we have

$$\frac{\Psi^2(q)\chi(q^2)}{\chi^2(-q)} = \sum_{n=0}^{\infty} \frac{q^n}{1 + q^{4n+1}} + \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1 + q^{4n+3}} \quad (2.4)$$

Further, replacing q by q^4 and then taking $\alpha = -q$ and $z = q^3$ in (2.1), we get

$$2\Psi(q^2)\Psi(q^4) = \sum_{n=0}^{\infty} \frac{q^{3n}}{1 + q^{4n+1}} + \sum_{n=0}^{\infty} \frac{q^n}{1 + q^{4n+3}} \quad (2.5)$$

Now, if we replace q by q^2 and then set $\alpha = -z = -q$ in (2.1), we get

$$\Psi^2(q^2) = \sum_{n=0}^{\infty} \frac{q^n}{1 + q^{2n+1}} \quad (2.6)$$

Again, replacing q by q^3 and setting $-\alpha = z = q$ in (2.1), we get

$$c(q) = 3 \sum_{n=0}^{\infty} \frac{q^{n+(1/3)}}{1 + q^{3n+1}} + 3 \sum_{n=0}^{\infty} \frac{q^{2n-(4/3)}}{1 + q^{3n+2}} \quad (2.7)$$

Further (2.1), with q, α, z replaced by $q^2, -1, q$, respectively, yields

$$\Phi^2(q) = 2 \sum_{n=0}^{\infty} \frac{q^n}{1 + q^{2n}} + 2 \sum_{n=0}^{\infty} \frac{q^{n+1}}{1 + q^{2n+2}} \quad (2.8)$$

Now, if we replace q by q^3 and set $\alpha = z = q$ again in (2.1), we get

$$c(q) = 3 \sum_{n=0}^{\infty} \frac{q^{n+(1/3)}}{1 - q^{3n+1}} - 3 \sum_{n=0}^{\infty} \frac{q^{2n+(4/3)}}{1 - q^{3n+2}} \quad (2.9)$$

Now, equations (2.7) and (2.9) lead to

$$c(q) = 3 \sum_{n=0}^{\infty} \frac{q^{n+(1/3)}}{1 - q^{6n+2}} - 3 \sum_{n=0}^{\infty} \frac{q^{5n+(10/3)}}{1 - q^{6n+4}} \quad (2.10)$$

Next, replacing q by q^3 and then taking $\alpha = q$ in (2.2), we get

$$c^2(q) = 9 \sum_{k=0}^{\infty} \frac{(k+1)q^{k+(2/3)}}{1 - q^{3k+2}} + 9 \sum_{k=0}^{\infty} \frac{kq^{2k+(2/3)}}{1 - q^{3k+1}} \quad (2.11)$$

Further, replacing q by q^2 and then setting $\alpha = q$ in (2.2), we get

$$\Psi^4(q) = \sum_{k=0}^{\infty} \frac{(2k+1)q^k}{1 - q^{2k+1}} \quad (2.12)$$

Also, with $\alpha = -q$ in (2.2), we get

$$\Phi^4(-q) = 4 \sum_{n=-\infty}^{\infty} \frac{(-)^k(k+1)q^k}{1 + q^k} \quad (2.13)$$

Further, replacing q by $-q$ and setting $\alpha = -q$ in (2.2), we get

$$\Psi^4(-q) = \sum_{k=0}^{\infty} \frac{(2k+1)(-q)^k}{1 + q^{2k+1}} \quad (2.14)$$

which is same as (2.12) with q replaced by $-q$.

Comparing (2.6) and (2.12) we have

$$\sum_{n=0}^{\infty} \frac{(2n+1)q^{2n}}{1 - q^{4n+2}} = \left\{ \sum_{n=0}^{\infty} \frac{q^n}{1 + q^{2n+1}} \right\}^2 \quad (2.15)$$

from (2.8) and (2.13) we get,

$$\sum_{n=-\infty}^{\infty} \frac{(n+1)q^n}{1 + (-q^n)} = \left\{ \sum_{n=-\infty}^{\infty} \frac{q^n}{1 + q^{2n}} \right\}^2 \quad (2.16)$$

A number of similar other results can also be scored.

References

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