

Linear Generating Relations and Hypergeometric Transformations

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Abstract:

The main object of this paper is to obtain four linear generating relations for generalized Hermite polynomials of Dickinson-Warsi and two hypergeometric transformations for Gauss's ordinary hypergeometric function, using Barr's identity and Carlitz's generating relations.

Keywords and Phrases:

Linear generating relations, Generalized (Associated or Sonine) Laguerre polynomials, Generalized Hermite polynomials, Hypergeometric transformations.

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1- Introduction and Preliminaries :

It is well known that the Hermite polynomials $H_n(x)$ can be entirely reduced to the associated Laguerre polynomials $L_n^{(a)}(x)$ with the parameters $a = \pm \frac{1}{2}$ by means of the formulas [7, p.216(Q.1); 9, p.106(5.6.1)]

$$H_{2n}(x) = (-1)^n 2^{2n} (n)! L_n^{(-\frac{1}{2})}(x^2) \quad (1.1)$$

$$H_{2n+1}(x) = (-1)^n 2^{2n+1} (n)! x L_n^{(\frac{1}{2})}(x^2) \quad (1.2)$$

This relationship of Hermite to Laguerre polynomials is generalized by Dickinson and Warsi [4,p.258(10),p.259(12); see also 6, p. 352(4.1, 4.2)] in the form

$$H_{2n}^{(a)}(x) = (-1)^n 2^{2n} (n)! L_n^{(a)}(x^2) \quad (1.3)$$

and

$$H_{2n+1}^{(a)}(x) = (-1)^n 2^{2n+1} (n)! x L_n^{(a+1)}(x^2) \quad (1.4)$$

In 1960, Carlitz [2,pp.221-222(8,10); see also 8,p.420(20,21); 5,p.58] gave the following two generating relations

$$\sum_{n=0}^{\infty} \frac{(m+n)!}{m! n!} L_{n+m}^{(a)}(x) t^n = (1-t)^{-1-a-m} \exp\left[\frac{-xt}{1-t}\right] L_m^{(a)}\left[\frac{x}{1-t}\right] \quad (1.5)$$

$$\sum_{n=0}^{\infty} \frac{(m+n)!}{m! n!} L_{m+n}^{(a-n)}(x) t^n = (1+t)^a \exp[-xt] L_m^{(a)}[x(1+t)] \quad (1.6)$$

Throughout in present paper, we use the following standard notations:

$\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1, -2, -3, \dots\} = \mathbb{Z}_0^- \setminus \{0\}$.

Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ ($\lambda, \nu \in \mathbb{C}$) is defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1) \dots (\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases}$$

it being understood *conventionally* that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

In 1969, Barr [1,p.591(1)] gave the following hypergeometric series identity

$$\begin{aligned}
{}_A F_B \left[\begin{matrix} (a_A); \\ (b_B); \end{matrix} z \right] &= {}_{2A} F_{2B+1} \left[\begin{matrix} \frac{(a_A)}{2}, \frac{(a_A)+1}{2}; \\ \frac{1}{2}, \frac{(b_B)}{2}, \frac{(b_B)+1}{2}; \end{matrix} 4^{(A-B-1)} z^2 \right] + \\
&\quad z \frac{\prod_{m=1}^A (a_m)}{\prod_{n=1}^B (b_n)} {}_{2A} F_{2B+1} \left[\begin{matrix} \frac{(a_A)+1}{2}, \frac{(a_A)+2}{2}; \\ \frac{3}{2}, \frac{(b_B)+1}{2}, \frac{(b_B)+2}{2}; \end{matrix} 4^{(A-B-1)} z^2 \right] \quad (1.7)
\end{aligned}$$

where, for convergence, $A \leq B$ and $|z| < \infty$, or $A = B + 1$ and $|z| < 1$, it being assumed that $b_n \neq 0, -1, -2, -3, -4, \dots \quad \forall \quad n \in \{1, 2, 3, \dots, B\}$ and (a_A) represents the sequence of parameters given by $a_1, a_2, a_3, \dots, a_A$.

It is to be noted that linear generating relations (2.1),(2.2),(2.3),(2.4) for even and odd generalized Hermite polynomials of Dickinson-Warsi and two hypergeometric transformations (2.5),(2.6) do not seem to have been recorded earlier.

2- Main Linear Generating Relations and Hypergeometric Transformations :

Any values of parameters and variables leading to the results given in this section which do not make sense, are tacitly excluded.

$$\sum_{n=0}^{\infty} H_{2n+2m}^{(a-n)}(x) \frac{t^n}{n!} = \frac{(1-4t)^{(a-\frac{1}{2})}}{2x} \exp[4x^2 t] H_{2m+1}^{(a-1)} \left[x \sqrt{(1-4t)} \right] \quad (2.1)$$

$$\sum_{n=0}^{\infty} H_{2n+2m+1}^{(a-n)}(x) \frac{t^n}{n!} = 2x(1-4t)^{(a+1)} \exp[4x^2 t] H_{2m}^{(a+1)} \left[x \sqrt{(1-4t)} \right] \quad (2.2)$$

$$\sum_{n=0}^{\infty} H_{2n+2m}^{(a)}(x) \frac{t^n}{n!} = \frac{(1+4t)^{-(a+m+\frac{1}{2})}}{2x} \exp \left[\frac{4x^2 t}{1+4t} \right] H_{2m+1}^{(a-1)} \left[\frac{x}{\sqrt{(1+4t)}} \right] \quad (2.3)$$

$$\sum_{n=0}^{\infty} H_{2n+2m+1}^{(a)}(x) \frac{t^n}{n!} = 2x(1+4t)^{-(a+m+2)} \exp \left[\frac{4x^2 t}{1+4t} \right] H_{2m}^{(a+1)} \left[\frac{x}{\sqrt{(1+4t)}} \right] \quad (2.4)$$

$$\begin{aligned}
& {}_2F_1 \left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}; \\ \frac{1}{2} \end{matrix}; \frac{4x^2y}{(1+y)^2} \right] \\
&= (1+y)^a {}_2F_1 \left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}; \\ \frac{1}{2} \end{matrix}; \left(x + \sqrt{x^2-1}\right)^2 y \right] {}_2F_1 \left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}; \\ \frac{1}{2} \end{matrix}; \left(x - \sqrt{x^2-1}\right)^2 y \right] \\
&+ y(1+y)^a a^2 {}_2F_1 \left[\begin{matrix} \frac{a+1}{2}, \frac{a+2}{2}; \\ \frac{3}{2} \end{matrix}; \left(x + \sqrt{x^2-1}\right)^2 y \right] {}_2F_1 \left[\begin{matrix} \frac{a+1}{2}, \frac{a+2}{2}; \\ \frac{3}{2} \end{matrix}; \left(x - \sqrt{x^2-1}\right)^2 y \right] \quad (2.5)
\end{aligned}$$

$$\begin{aligned}
& {}_2F_1 \left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}; \\ \frac{3}{2} \end{matrix}; \frac{4x^2y}{(1+y)^2} \right] \\
&= \frac{(1+y)^a \left(x + \sqrt{x^2-1}\right)}{2x} {}_2F_1 \left[\begin{matrix} \frac{a-1}{2}, \frac{a}{2}; \\ \frac{1}{2} \end{matrix}; \left(x - \sqrt{x^2-1}\right)^2 y \right] {}_2F_1 \left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}; \\ \frac{3}{2} \end{matrix}; \left(x + \sqrt{x^2-1}\right)^2 y \right] \\
&+ \frac{(1+y)^a \left(x - \sqrt{x^2-1}\right)}{2x} {}_2F_1 \left[\begin{matrix} \frac{a-1}{2}, \frac{a}{2}; \\ \frac{1}{2} \end{matrix}; \left(x + \sqrt{x^2-1}\right)^2 y \right] {}_2F_1 \left[\begin{matrix} \frac{a}{2}, \frac{a+1}{2}; \\ \frac{3}{2} \end{matrix}; \left(x - \sqrt{x^2-1}\right)^2 y \right] \quad (2.6)
\end{aligned}$$

Provided that $\left| \frac{x^2y}{(1+y)^2} \right| < \frac{1}{4}$

3- Proofs of Linear Generating Relations and Hypergeometric Transformations :

Consider the series

$$S_1 = \sum_{n=0}^{\infty} H_{2n+2m}^{(a-n)}(\sqrt{x}) \frac{t^n}{n!} \quad (3.1)$$

Using the result(1.3), we get

$$S_1 = (-1)^m 2^{2m} \sum_{n=0}^{\infty} (m+n)! L_{m+n}^{(a-n)}(x) \frac{(-4t)^n}{n!} \quad (3.2)$$

Now using the result (1.6) in (3.2), we get

$$S_1 = (-1)^m 2^{2m} m! (1-4t)^a \exp[4xt] L_m^{(a)}[x(1-4t)] \quad (3.3)$$

Now applying (1.4) and replacing x by x^2 , we get the generating relation (2.1).

If we apply same technique for the series

$$S_2 = \sum_{n=0}^{\infty} H_{2n+2m+1}^{(a-n)} (\sqrt{x}) \frac{t^n}{n!} \quad (3.4)$$

we get the generating relation (2.2).

Now consider the another series

$$S_3 = \sum_{n=0}^{\infty} H_{2n+2m}^{(a)} (\sqrt{x}) \frac{t^n}{n!} \quad (3.5)$$

Using the results (1.3) and (1.5) in equation(3.5), we get

$$S_3 = (-1)^m 2^{2m} m! (1+4t)^{-1-a-m} \exp \left[\frac{4xt}{1+4t} \right] L_m^{(a)} \left[\frac{x}{1+4t} \right]$$

Now using the result (1.4) and replacing x by x^2 , we get the generating relation (2.3).

Similarly we obtain the generating relation (2.4) from the following series

$$S_4 = \sum_{n=0}^{\infty} H_{2n+2m+1}^{(a)} (\sqrt{x}) \frac{t^n}{n!} \quad (3.6)$$

Now consider the following identity

$$(1+y^2)^{-a} \left[1 - \frac{2xy}{1+y^2} \right]^{-a} = \left[1 - (x + \sqrt{x^2-1})y \right]^{-a} \left[1 - (x - \sqrt{x^2-1})y \right]^{-a}$$

Writing $\left[1 - \frac{2xy}{1+y^2} \right]^{-a}$, $\left[1 - (x + \sqrt{x^2-1})y \right]^{-a}$ and $\left[1 - (x - \sqrt{x^2-1})y \right]^{-a}$ in hypergeometric notations, using Barr's identity (1.7), replacing y by $i\sqrt{y}$, equating real and imaginary parts and making suitable adjustment of parameters, we get the hypergeometric transformations (2.5) and (2.6). For similar method, readers are advised to refer the paper [3].

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