

On Certain Transformations of Basic Hypergeometric Function using q-Fractional Operators

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Abstract: The object of this paper is to derive a transformation expressing a basic hypergeometric function in terms of a finite sum of basic hypergeometric function by the application of q-fractional derivatives using some known results.

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1. Introduction

For real or complex a , $q < 1$, the q -shifted factorial is defined by

$$(a, q)_n = \begin{cases} 0 & \text{if } n = 0; \\ (1-a)(1-aq)(1-aq^2)\dots, (1-aq^{n-1}) & \text{if } n \in N. \end{cases} \quad (1.1)$$

The generalized basic hypergeometric series (cf. Gasper and Rahman[1]) is defined by

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r)_n}{(b_1, b_2, \dots, b_s)_n} [(-1)^n q^{n(n-1)/2}]^{1+s-r} z^n, \quad (1.2)$$

where r and s are positive integers, $q \neq 0$ when $r > s + 1$, the numerator parameters $a_1; \dots; a_r$ and the denominator parameters $b_1; \dots; b_s$ being complex quantities provided that $b_j \neq q^{-m}$; $m = 0; 1; \dots; j = 1; 2; \dots; s$: If $0 < |q| < 1$, the above series converges absolutely for all x if $r \leq s$ and for $|x| < 1$ if $r = s + 1$. This series also converges absolutely if $|q| > 1$ and $|z| < |b_1 b_2 \dots b_s| / |a_1 a_2 \dots a_r|$.

The abnormal type of generalized basic hypergeometric series ${}_r\phi_s(\cdot)$ is defined as

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r; q, z \\ b_1, b_2, \dots, b_s; q^\lambda \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_n}{(b_1, b_2, \dots, b_s; q)_n} z^n q^{\lambda n(n+1)/2}, \quad (1.3)$$

where $\lambda > 0$ and $|q| < 1$.

Jain[3] defined the generalized basic hypergeometric function of two variables as:

$$\begin{aligned} & \Phi_{C:D';D''}^{A:B';B''} \left[\begin{matrix} (a) : (b'); (b''); q, x, y \\ (c) : (c'); (c''); i, j, k \end{matrix} \right] \\ &= \sum_{m,n=0}^{\infty} \frac{\Pi_{t=1}^A(a_t; q)_{m+n} \Pi_{t=1}^{B'}(b'_t; q)_m \Pi_{t=1}^{B''}(b''_t; q)_n}{\Pi_{t=1}^C(c_t; q)_{m+n} \Pi_{t=1}^{D'}(d'_t; q)_m \Pi_{t=1}^{D''}(d''_t; q)_n} \\ & \quad \times \frac{x^n y^n q^{im(m-1)/2 + in(n-1)/2 + kmn}}{(q; q)_m (q; q)_n}, \end{aligned} \quad (1.4)$$

where $|x| < 1, |y| < 1, 0 < |q| < 1$, for special values of i, j, k and A, B', B'', C, D', D'' are non negative integers and (a) denotes the sequence of A parameters $a_1, a_2, \dots, a_A$. When $i = j = k = 0$, the first member of above equation is written as

$$\Phi_{C:D';D''}^{A:B';B''} \left[\begin{matrix} (a) : (b'); (b''); \\ (c) : (c'); (c''); \end{matrix} \begin{matrix} q, x, y \end{matrix} \right]$$

Agrawal[1] introduced the fractional q -derivative operator of order β in terms of q -integral as:

$$D_{z,q}^{\beta} f(z) = \frac{1}{\Gamma_q(-\beta)} \int_0^z (z - tq)_{-\beta-1} f(t) d(t; q). \quad (1.5)$$

Using the q -integral [3, (1.11.3), p.19], the above operator can written as:

$$D_{z,q}^{\beta} f(z) = \frac{z^{-\beta}(1-q)}{\Gamma_q(-\beta)} \sum_{k=0}^{\infty} q^k (1 - q^{k+1})_{-\beta-1} f(zq^k). \quad (1.6)$$

For $f(z) = z^{\alpha-1}$ the above equation can be written as:

$$D_{z,q}^{\beta} \{z^{\alpha-1}\} = \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha - \beta)} z^{\alpha-\beta-1}, \quad \text{Re } \alpha > 0, \quad \forall \beta \quad (1.7)$$

For arbitrary α , the q -binomial coefficient [3, I.42, p.235] is defined by

$$\left[\begin{matrix} \alpha \\ n \end{matrix} \right]_q = \frac{(q^{-\alpha}; q)_n}{(q; q)_n} (-q^{\alpha})^n q^{-\binom{n}{2}} \quad (1.8)$$

Yadav, Purohit and Vyas [4,(20), p.128] shown that a series can be expressed as a sum of fractional q-derivative operator namely

$$\sum_{n=0}^{\infty} \frac{(\alpha q^{\beta}; q)_n}{(\alpha; q)_n} A_n z^n = \sum_{n=0}^{\infty} \left[\begin{matrix} \alpha \\ n \end{matrix} \right]_q q^{n^2+n\alpha-n} \frac{z^n (1-q)^n}{(\alpha; q)_n} D_{z,q}^n \{f(z)\}, \quad (1.9)$$

where $Re(\alpha q^{\beta}) > 0$, the arbitrary sequence of real and complex number is bounded in the domain $|z| < 1$.

Let us define a basic regular function $f(z)$ in terms of power series with a sequence $\{A_n\}$ of arbitrary coefficients,

$$f(z) = \sum_{n=0}^{\infty} A_n z^n \quad (1.10)$$

The fractional q-derivative of generalized basic hypergeometric function due to Yadav and Purohit [4] is given by

$$\begin{aligned} & D_{z,q}^n \left\{ {}_r\Phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} ; q, z \right] \right\} \\ &= \frac{(1-q)^{-n} (a_1, a_2, \dots, a_r; q)_n}{(b_1, b_2, \dots, b_s)_n} {}_r\Phi_s \left[\begin{matrix} a_1 q^n, a_2 q^n, \dots, a_r q^n \\ b_1 q^n, b_2 q^n, \dots, b_s q^n \end{matrix} ; q, z \right]. \end{aligned} \quad (1.11)$$

I will also use the following Transformation formula in our proof:

$${}_4\Phi_3 \left[\begin{matrix} a, b, \sqrt{ab}, -\sqrt{ab} \\ ab, \sqrt{abq}, -\sqrt{abq} \end{matrix} ; q, z \right] = {}_2\phi_1 \left[\begin{matrix} a, b \\ abq \end{matrix} ; q^2, z \right] {}_2\phi_1 \left[\begin{matrix} a, b \\ abq \end{matrix} ; q^2, qz \right] \quad (1.12)$$

2. Main Results

$$\begin{aligned} & {}_2\phi_1 \left[\begin{matrix} a, b \\ abq \end{matrix} ; q^2, z \right] {}_2\phi_1 \left[\begin{matrix} a, b \\ abq \end{matrix} ; q^2, qz \right] \\ &= \sum_{n \geq 0} \left(\begin{matrix} \beta \\ n \end{matrix} \right)_q q^{n(n+\alpha-1)} \frac{z^n (1-q)^n (a, b, \sqrt{ab}, -\sqrt{ab}; q)_n}{(ab, \sqrt{abq}, -\sqrt{abq}, \alpha q^{\beta}; q)_n} \times \end{aligned}$$

$${}_5\phi_4 \left[\begin{matrix} aq^n, bq^n, \sqrt{ab}q^n, -\sqrt{ab}q^n, \alpha q^n \\ abq^n, \sqrt{ab}q^{n+1/2}, -\sqrt{ab}q^{n+1/2}, \alpha q^{n+\beta} \end{matrix} ; q, z \right] \quad (2.1)$$

$$\begin{aligned} & {}_2\phi_1 \left[\begin{matrix} a^2, a^2q \\ a^4 \end{matrix} ; q^2, z \right] {}_2\phi_1 \left[\begin{matrix} a^2, a^2q \\ a^4 \end{matrix} ; q^2, z \right] \\ &= \sum_{n \geq 0} \binom{\beta}{n}_q q^{n(n+\alpha-1)} \frac{z^n (1-q)^n (ab, -ab, ab\sqrt{q}, -ab\sqrt{q}; q)_n}{(a^2b^2, -a^2, -b^2q, \alpha q^\beta; q)_n} \times \\ & {}_5\phi_4 \left[\begin{matrix} aq^n, bq^n, \sqrt{ab}q^n, -\sqrt{ab}q^n, \alpha q^n \\ abq^n, \sqrt{ab}q^{n+1/2}, -\sqrt{ab}q^{n+1/2}, \alpha q^{n+\beta} \end{matrix} ; q, z \right] \quad (2.2) \end{aligned}$$

3. Proof of (2.1) :

Let us choose an arbitrary sequence $\{A_n\}$ given by:

$$A_n = \frac{(a, b, \sqrt{ab}, -\sqrt{ab}, \alpha; q)_n}{(ab, \sqrt{ab}q, -\sqrt{ab}q, \alpha q^\beta; q)_n}$$

Using the above equation in (1.9) and then using the equation (1.10), we get the following:

$$\begin{aligned} & \sum_{n \geq 0} \frac{(a, b, \sqrt{ab}, -\sqrt{ab}; q)_n}{(ab, \sqrt{ab}q, -\sqrt{ab}q, q; q)_n} z^n \\ &= \sum_{n \geq 0} \binom{\beta}{n}_q q^{n(n+\alpha-1)} \frac{z^n (1-q)^n}{(\alpha; q)_n} \times \\ & D_{z,q}^n \left\{ \sum_{n \geq 0} \frac{(a, b, \sqrt{ab}, -\sqrt{ab}, \alpha; q)_n z^n}{(ab, \sqrt{ab}q, -\sqrt{ab}q, \alpha q^\beta; q)_n} \right\} \\ &= \sum_{n \geq 0} \binom{\beta}{n}_n q^{n(n+\alpha-1)} \frac{z^n (1-q)^n}{(\alpha; q)_n} \times \\ & D_{z,q}^n \left\{ {}_5\phi_4 \left[\begin{matrix} a, b, \sqrt{ab}, -\sqrt{ab}, \alpha \\ ab, \sqrt{ab}q, -\sqrt{ab}q, \alpha q^\beta \end{matrix} ; q, z \right] \right\} \end{aligned}$$

Now applying fractional q-derivative of generalized hypergeometric series due to Yadav, Purohit and Vyas [7, (20), p.128], we get:

$$\begin{aligned}
 & {}_4\Phi_3 \left[\begin{matrix} a, b, \sqrt{ab}, -\sqrt{ab} \\ ab, \sqrt{abq}, -\sqrt{abq} \end{matrix} ; q, z \right] \\
 &= \sum_{n \geq 0} \left(\begin{matrix} \beta \\ n \end{matrix} \right)_q q^{n(n+\alpha-1)} \frac{(z(1-q))^n (\alpha, a, b, \sqrt{ab}, -\sqrt{ab})_n}{(ab, \sqrt{abq}, -\sqrt{abq}, \alpha q^\beta; q)_n} \times \\
 & {}_5\phi_4 \left[\begin{matrix} aq^n, bq^n, \sqrt{ab}q^n, -\sqrt{ab}q^n, \alpha q^n \\ abq^n, \sqrt{ab}q^n, -\sqrt{ab}q^n, \alpha q^{\beta+n} \end{matrix} ; q, z \right]
 \end{aligned}$$

This completes the proof of the above result (2.1) using transformation formula(1.12) on the left hand side .

Proof of(2.2): Let us choose an arbitrary sequence $\{A_n\}$ given by:

$$A_n = \frac{(ab, -ab, ab\sqrt{q}, -ab\sqrt{q}, \alpha; q)_n}{(a^2b^2, -a^2, -b^2q, \alpha q^\beta, q; q)_n}$$

Using the above equation in (1.9)and then using the equation (1.10) and after doing some calculation we get the above result(2.2).

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